# MEET THE SURREAL NUMBERS

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# 1 INTRODUCTION

GH Hardy wrote in [1],"A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent that theirs, it is because they are made with ideas." And, I would add, because they are less culturally dependent. I heard a radio play long ago, possibly around the time of the Cuban missile crisis in 1962, about a world in which many millions of years after a nuclear war wiped out humanity, a species of intelligent lizard evolved, whose archeologists eventually unearthed remains of human civilisation. Would the lizards appreciate Mozart, Rembrandt or Shakespeare? Probably not. But would the mathematicians amongst them appreciate Conway's surreal numbers - yes I'm pretty sure they would, unless of course they had already discovered them themselves, in which case they would be very remarkable lizards indeed. Hardy also wrote that "The mathematician's patterns, like the painter's or the poet's, must be beautiful ... There is no permanent place in the world for ugly mathematics." The surreal numbers must be amongst the most beautiful patterns that mankind has yet produced, and that is why everyone who possibly can should study them.

[2] tells us what people have said about the surreal numbers. Conway himself said "I walked around for about six weeks after discovering the surreal numbers in a sort of permanent day-dream, in danger of being run over." This sense of reverie overtakes others who study them. Martin Kruskal, a mathematician of wide-ranging achievements, spent some of his later years studying the surreal numbers, and he wrote "The usual numbers are very familiar, but at root they have a very complicated structure. Surreals are in every logical, mathematical and aesthetic sense better." Of the quite magical way in which the numbers are created, Martin Gardner wrote "An empty hat rests on a table made of a few axioms of standard set theory. Conway waves two simple rules in the air, then reaches into almost nothing and pulls out an infinitely rich tapestry of numbers."

The surreal numbers form a field, which is to say that they can be added, subtracted, multiplied and divided, so long as you do not try to divide by 0. The include the familiar real numbers as a tiny subfield, and like the reals they are a lineally ordered field. So far so unremarkable, but they also include the transfinite ordinals, and since they are a field they include, along with the first infinite ordinal $\omega$, such wonders as $\omega - 1$, not to mention $\omega/2$ and $2/\omega$, and in fact they also include $\sqrt{\omega}$ and indeed $\omega^r$ for any real number, and for any surreal number.

These notes will not get you far into the theory. They were written as a handout to accompany a one hour talk at the 2017 annual conference of the Mathematical Association, and are an attempt to do properly what such a short talk can only hint at. Most of the results are proved, because the way proof works in this amazing world is one of its great beauties. Read and enjoy. If you find any mistakes, please let me know at jimsimonsfoxcote@googlemail.com.

# 2 PRELIMINARIES

## 2.1 Set Theory

We use sets all the time, but mostly without giving much thought to set theory and its axioms. Indeed I expect that there are plenty of us who get through life without realising that there are axioms for set theory. We use what is rather rudely called naive set theory, which means calling any collection of things, however defined, a set. That works just fine for almost all of mathematics and its applications. However, it won't quite do for the study of surreal numbers, so we shall have to look into set theory just a bit.

 

The reason naive set theory is inadequate is Russell's paradox . Let $S$ be the set of all sets that do not belong to themselves:

$$S = \{x : x \notin x\}$$

(I use a colon for "such that" in the definition of a set rather than a vertical line for a reason that will become clear later.) A little thought reveals that if $S$ is a member of itself then it isn't, and if it isn't then it is! So suddenly, when this was discovered, the foundations were washed away from under the whole of mathematics, which now appeared to be built upon a contraction, and therefore to be invalid. Except that nothing really changed, because of course mathematics is not really built on set theory: it had been going on for millennia perfectly happily before anyone even thought of set theory. It is more a case of set theory being bolted onto the bottom of mathematics. Anyway set theorists beavered away to create axioms for a theory of sets that could be used without contradiction, and the key idea is that there are some collections of things that do not count as sets. We shall be using an informal version of NBG set theory (von Neumann-Bernays-Gödel). This allows us to talk about classes, only some of which are sets. Classes that are not sets are called proper classes, and roughly speaking these are classes that are too big to be sets, such as the class of all sets. There is an axiom that no set can be a member of itself, and the key distinction between sets and proper classes is that only sets can be members of anything (sets or proper classes). It is as though the net constructed of { and } catches all the little fish, but the big ones slip through! The $S$ we defined above is now a proper class (in fact equal to the class of all sets), so cannot be a member of anything, least of all itself.

It is now a standard theorem of set theory, which we shall be using later, that there cannot be an infinite descending membership chain: $\cdots e \in d \in c \in b \in a$. There certainly can be infinite ascending membership chains: $a \in b \in c \in d \in \cdots$, and we shall be meeting lots of those. There are lots of other axioms, but all we need to know for what we might call not-quite-so-naive set theory is that the class of natural numbers is a set, and we can form sets from other sets by taking unions, power sets and sensibly defined subsets.

## 2.2 Ordinals

Just as we don't need much set theory, we don't need to know much about ordinals, but it is helpful to know a little. Ordinals extend the idea of counting into the infinite in the simplest way imaginable: just keep on counting. So we start with the natural numbers: $0, 1, 2, 3, 4, \ldots$, but we don't stop there, we keep on with a new number called $\omega$, then $\omega + 1$, $\omega + 2$, $\omega + 3$ and so on. After all those we come to $\omega + \omega = \omega.2$. Carrying on we come to $\omega.3$, $\omega.4$ etc and so on to $\omega^2$. Carrying on past things like $\omega^2.7 + \omega.42 + 1$, we'll come to $\omega^3$, $\omega^4$ and so on to $\omega^\omega$, and this is just the beginning. To see a bit more clearly where this is heading, we'll look at von Neumann's construction of the ordinals.

**The Ordinal Construction.** *An ordinal is the set of all previously defined ordinals.*

That's it! This of course is a recursive definition, and to start with we have no ordinals defined, but we do have a set of ordinals, the empty set, so this is our first ordinal, which we call 0. Then we can start making more ordinals:

$$
\begin{aligned}
0 &\equiv \{\,\} \\
1 &\equiv \{0\} \\
2 &\equiv \{0,1\} \\
3 &\equiv \{0,1,2\} \\
&\vdots \\
\omega &\equiv \{0,1,2,3,\ldots\} \\
\omega+1 &\equiv \{0,1,2,3,\ldots,\omega\}
\end{aligned}
$$

and so on, where we are using $\equiv$ to assign a name to an ordinal. (I use $\{\,\}$ for the empty set rather than $\emptyset$ for a reason that will become clear later.) These ordinals only have countably many members, and so are said to be countable ordinals, as are $\omega^\omega$, $\omega^{\omega^\omega}$, $\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}$ and many more wonders beyond that, but eventually, the set of all countable ordinals defines, in fact *is*, the first uncountable ordinal, after which of course we just keep on counting. The class of all ordinals, called **On**, is too big to be a set, and therefore is not itself an ordinal (if it were a set, it would be a member of itself, which is a contradiction.) The next theorem has the key results about ordinals. Ordinal-valued variables will be lower case greek letters, except $\beta$ and $\kappa$ for which special meanings will be introduced later.

**The Ordinal Theorem.** (I prefer named theorems to numbered ones: who can remember what Theorem 17 said when it is referred to later?)

*(a) Every set of ordinals has a least member, equivalently every descending sequence of ordinals is finite (ie they are "well ordered");*

*(b) Every ordinal $\alpha$ has a successor $\alpha+1$.*

*(c) For every set $\Gamma$ of ordinals, there is an ordinal* $\sup\Gamma$ *that is the smallest that is at least as large as any of them.*

*Proof.*

(a) Take their intersection.

(b) Take $\alpha \cup \{\alpha\}$.

(c) $\cup\Gamma$ is an ordinal, and is the least ordinal that is at least as large as all the original ones, so $\sup\Gamma = \cup\Gamma$.

Authors often use some sort of right-justified blob, rather morbidly called a tombstone, to signal the end of a proof. I shall use this rather more joyful symbol: ☑

Not all ordinals have predecessors. Those that have are called successor ordinals. Those that do not, apart from 0, are called limit ordinals. This conjures up the image of $\omega$ is a sort of accumulation point just beyond the end of the finite ordinals. It will turn out that for surreal ordinals, that image is utterly wrong.

There are arithmetic operations for ordinals, which, like the ordinals themselves, are defined recursively. They have been used earlier in this subsection to give names to the examples of ordinals, but we shall have no further use for them, and shall not be defining them, or establishing any if their properties, except that we shall refer to the successor of an ordinal $\alpha$ as $\alpha + 1$, and a predecessor of an ordinal that has one (ie neither 0 nor a limit ordinal) as $\alpha - 1$. The von Neumann construction itself and the Ordinal Theorem contain all that we need, and we only need that because we shall be finding a copy of the ordinals inside the surreal numbers, and we need to be able to recognise them. [3] is an excellent introduction to ordinals for those who want a little more detail.

# 3 CONSTRUCTING THE SURREAL NUMBERS

In [4] Conway gives the still definitive formulation of the surreal numbers (henceforth just numbers). Other authors have developed the theory from different starting points, which we shall briefly look at later, (eg [5], [8]), but here we stick to Conway's approach.

## 3.1 The Construction

The basic tool for constructing surreal numbers is the double set, ie an ordered pair of sets. (Conway calls these double sets games, for reasons that need not detain us.). Conway uses the notation $\{\dots \mid \dots\}$ where what goes in the spaces are either the names of the sets, or the elements of the sets. (The reason for using a colon in the middle of set definitions is to avoid confusion with this vertical line.) Elements of the left-hand set are called left options, and elements of the right-hand set are called right options, and these options will always turn out to be numbers, so that just as ordinals are sets of ordinals, numbers are double sets of numbers. Typical left and right options of a double set $x$ are written as $x^L$ and $x^R$, and a typical option, if we don't care whether it is left or right, is $x^O$ (that's a capital O, and cutely, O is midway between L and R in the alphabet). We shall use the symbol $\varepsilon$ to mean "is an option of", similar to, but I hope distinguishable from, the familiar set theoretic symbol $\in$ meaning "is a member of". Now, since the surreal numbers are indeed numbers, it is not giving too much away to reveal that there are surreal numbers called 0, 1 and 2, and so, to give an example, the left-hand set might be $L = \{0, 1\}$, so that the left options are 0 and 1, and the right-hand set might be $R = \{2\}$, so that the only right option is 2. The double set can then written either as $\{0, 1 \mid 2\}$, or as $\{L \mid R\}$. Actually we can be quite cavalier about this, allowing mixed usages such as $\{L \mid 2\}$. This is not ambiguous as, unlike with ordinals, no set of numbers is itself a number. To help with keeping this clear, sets of numbers will always be given upper case names, whereas numbers will be given lower case names (or proper number names such as 42). Now, here is the construction:

**The Surreal Construction.** *A surreal number is a double set $x$ of numbers such that $x^R \leq x^L$ never holds. Double sets $x$ and $y$ satisfy $x \leq y$ if and only if neither $y \leq x^L$ nor $y^R \leq x$ ever hold. $y$*

Just to be completely clear, "$x^R \leq x^L$ never holds" means that there is no right option of $x$ that is $\leq$ any left option. Before diving in to see how this works, let's step back and look at the big picture.

- Like von Neumann's construction of the ordinals this construction is recursive, starting from nothing, but this time there are two linked recursion: we can't construct a number until we have established the truth or otherwise of the $\leq$ relationship between its options, and obviously we can't establish whether $\leq$ holds between two numbers until we have constructed them.

 

- The relation $\leq$ is defined for double sets, not just for numbers. All the double sets we meet will in fact be numbers, but sometimes it will be convenient to be able to establish inequalities before proving that they are numbers.

- The symbol $\leq$ makes it look as though this relationship is some sort of order, and indeed it will turn out to be a total order, but we can't assume that. We are going to have to prove it, but before getting there, let's look at the motivation for the definition. The idea is to create a total order with the property that $x^L < x < x^R$, so that the new number fits in the interval between all its left options and all it right options, rather like Dedekind sections. Then if $x$ is to be $\leq y$, we cannot have, for example $y^R \leq x$, for then we should have $y^R \leq x \leq y < y^R$. So the conditions in the definition of $\leq$ are the weakest possible to create an order of the sort we want we want, and it is remarkable that it does indeed create such an order. We shall see that again: very weak definitions pulling off remarkable feats.

- This construction builds all the numbers in one go, but there is no arithmetic here, only an order. It builds negative numbers without any concept of addition or subtraction, and it builds fractions without any notion of multiplication or division. We shall add in those arithmetic operations later, but quite lot later: there is a lot of explore before we get there.

This situation contrasts markedly with the standard ways of constructing the real numbers, which goes something like this:

**The Real Construction.**

- *Construct the natural numbers, using von Neumann's ordinal construction, but stopping after the finite ordinals;*

- *Define addition and multiplication on the natural numbers, and prove that have the right properties;*

- *Define integers as equivalence classes of ordered pairs of natural numbers where* $(a, b) \sim (c, d)$ *iff* $a + d = c + b$*;*

- *Define addition and multiplication on these integers, and prove they have the right properties;*

- *Define rational numbers as equivalence classes of ordered pairs of integers of which the second is non-zero, where* $(a, b) \sim (c, d)$ *iff* $a \times d = c \times b$*;*

- *Define addition and multiplication on these rationals, and prove they have the right properties;*

- *Define real numbers as Dedekind sections of the rationals;*

- *Define addition and multiplication on these reals, and prove they have the right properties.*

That's an eight stage process with four different sets of arithmetic operations. Well, as I said, most of mathematics is not really built on set theory: this construction is an ugly kluge bolted onto the bottom of the real numbers that got along just fine for 2000 years without it. The surreals are different though; they really are built on set theory.

So let's see how all this works by constructing some numbers. To begin with we have no numbers, so the only available set of numbers is the empty set, $\{\,\}$, so the only double set is

$\{\,|\,\}$, (consistency here is the reason for writing the empty set as $\{\,\}$). We don't even have to look at the definition of $\leq$ to see that we never have a right option of this being $\leq$ a left option, because there are no options at all. So this is a number, which we shall call 0:

$$\{\,|\,\} \equiv 0$$

where, as with the ordinals, we use $\equiv$ to assign a name to a number. At the moment, 0 is just an arbitrary name, but later on, when we have defined addition, we shall be able to show that this is the right name, because this number will turn out to be the additive identity.

Whenever we construct numbers, we must check whether $\leq$ holds between any of them. That's pretty easy here, we just need to check whether $0 \leq 0$. Here is how we shall lay out questions like that, posing the question and expanding the numbers on the next line:

$$\begin{array}{ccccc} 0 & \leq & 0 & ? \\ \{\,|\,\} & & \{\,|\,\} & \end{array}$$

This will be true unless the right number is $\leq$ a left option of the left number, or a right option of the right number that is $\leq$ the left number. Clearly none of these things happen here, so yes $0 \leq 0$.

That's it for the first round of number creation. It will turn out that numbers have birthdays, which are ordinals. We'll define this properly later, when it will be clear that 0 is the only number born on day 0.

On to day 1. We now have a new set of numbers, $\{0\}$, and so three new double sets, $\{0 \mid 0\}$, $\{\,|\, 0\}$ and $\{0 \,|\,\}$. The first is not a number because, by what we have just shown, it does have a right option $\leq$ a left option. The other two are numbers because they don't have both sort of option. We shall give them names too:

$$\{\,|\, 0\} \equiv -1 \qquad \text{and} \qquad \{0 \,|\,\} \equiv 1$$

Again we shall be able to show later that these are the right names. Now we have to check $\leq$, first between new numbers and old ones, and then amongst the new ones.

$$\begin{array}{ccccc} 0 & \leq & 1 & ? \\ \{\,|\,\} & & \{0\,|\,\} & \end{array}$$

This is true because the right number has no right options, and the left number has no left options.

$$\begin{array}{ccccc} 1 & \leq & 0 & ? \\ \{0\,|\,\} & & \{\,|\,\} & \end{array}$$

This is false, because the right number is $\leq$ a left option of the left number. We are getting a hint now that $\leq$ is going to turn out to be an order relationship, so let's define the other order symbols:

$y \geq x$ means $x \leq y$;

$x < y$ means $x \leq y$ and $x \ngeq y$;

$y > x$ means $x < y$;

$x = y$ means $x \leq y$ and $x \geq y$.

With this notation, we have just shown that $0 < 1$, and $-1 < 0$ is just as easy.

$$\begin{array}{ccccc} 1 & \leq & 1 & ? \\ \{0\,|\,\} & & \{0\,|\,\} & \end{array}$$

This is the first case satisfied non-vacuously, because the left number does have a left option, but the right number is not $\leq$ it, by what we have just shown. So $1 \leq 1$ and indeed $1 = 1$. Of course $-1 = -1$, and the reader invited to prove that $-1 < 1$.

On day 2 the plot thickens a bit. We have three numbers, so eight sets of numbers, and 64 double sets, including the ones we've already seen. It's easy to winnow out the ones like $\{1 \mid 0\}$ that are not numbers, and we are left with 17 new numbers. Look at $\{-1 \mid 1\}$, and compare it with 0. We'll temporarily call it $x$:

$$\begin{array}{ccc} 0 & \leq & x \quad ? \\ \{\,\mid\,\} & & \{-1 \mid 1\} \end{array}$$

This is definitely true, as the right number's right option is bigger than 0, and the left number has no left option. Equally $x \leq 0$, so that means $x = 0$. So we have different numbers that are equal. We shouldn't be too alarmed about this, because we are used to it with fractions: $3/6 = 1/2$. Of course set theorists will tell you that a rational number is an equivalence class of fractions like this, but we don't really believe them. We behave as though $3/6$ and $1/2$ are distinct numbers, distinct because they have different numerators and denominators, but they are equal. In the same way, 0 and $\{-1 \mid 1\}$ are distinct, having different options, but they are equal. We shall return to this topic, but meanwhile we shall find that all the numbers we have created fit into eight equivalence classes of equal numbers.

$$\begin{array}{rlll}
-2 \equiv \{\,\mid -1, 0\} & = \{\,\mid -1, 0, 1\} & = \{\,\mid -1\} & = \{\,\mid -1, 1\} \\
-1 \equiv \{\mid 0\} & = \{\mid 0, 1\} & & \\
-1/2 \equiv \{-1 \mid 0\} & = \{-1 \mid 0, 1\} & & \\
0 \equiv \{\,\mid\,\} & = \{-1 \mid 1\} & = \{-1 \mid\,\} & = \{\,\mid 1\} \\
1/2 \equiv \{0 \mid 1\} & = \{-1, 0 \mid 1\} & & \\
1 \equiv \{0 \mid\} & = \{-1, 0 \mid\} & & \\
2 \equiv \{0, 1 \mid\,\} & = \{-1, 0, 1 \mid\,\} & = \{1 \mid\,\} & = \{-1, 1 \mid\,\}
\end{array}$$

There is a pattern to which number from each equivalence class gets a special name, such as $-2$ or $1/2$. In truth it doesn't much matter, since they are all equal, but there are style points to be had from an elegant system. All will be revealed in due course. Meanwhile, if you take any two numbers from different equivalence classes, you'll find that $<$ holds between them in exactly the way their names suggest. To check all these inequalities, you have first to check all the new numbers against 0, then against the day 1 numbers, $-1$ and 1, and finally against each other. So let's take $1/2$.

$$\begin{array}{ccc} 0 & \leq & 1/2 \quad ? \\ \{\,\mid\,\} & & \{0 \mid 1\} \end{array}$$

This is true. The right number has a right option, but it is greater than the left number.

$$\begin{array}{ccc} 1/2 & \leq & 0 \quad ? \\ \{0 \mid 1\} & & \{\,\mid\,\} \end{array}$$

This in not true, because the left number has a left option that is $\geq$ the right number, so $1/2 > 0$. Then it is a trivial but long-winded business to show in turn that $1/2 \leq 1$ and $1/2 \not\geq 1$ so that $1/2 < 1$, and that $1/2 \geq -1$ and $1/2 \not\leq -1$, so that $1/2 > -1$. Exactly the same inequalities hold for $y \equiv \{-1, 0, \mid 1\}$, and having established all those, we can shown that indeed $1/2 = y$. For example:

$$\begin{array}{ccc} 1/2 & \leq & y \quad ? \\ \{0 \mid 1\} & & \{-1, 0 \mid 1\} \end{array}$$

 

This is clearly true, since we have already established that $1/2 < 1$ and $0 < y$.

Having gone through a similar rigmarole for all the other new numbers, we can verify all the relationships between then, such as $1/2 < 2$.

On day 3, the new named numbers, to one of which all new numbers are equal, will be

$$-3, \quad -3/2, \quad -3/4, \quad -1/4, \quad 1/4, \quad 3/4, \quad 3/2, \quad 3$$

and so on through the finite days, expanding outwards and filling in the spaces, so that after all the finite days we shall have all the dyadic rationals. On day $\omega$ that process continues, producing all the other rationals and the real numbers in the spaces between the dyadic rationals, and $\omega$ itself off to the right, as well as some other surprises. The process doesn't stop there of course: ever more detail is filled in in the spaces, and the range of numbers continues to expand in a giant binary tree. We'll see the details later.

## 3.2 Getting the Numbers in Order

It is beginning to look as though $\leq$ is a proper order, so it is probably time to prove that. First we need:

**The Descending Chain of Options Theorem.** *Numbers can have no infinite descending chain of options.*

*Proof.* The cutest way to see that is to create sets alongside the numbers, ignoring the difference between left and right options, roughly speaking replacing the vertical line with a comma. Slightly more formally, for any number $x$, we create the set $f(x) = \{f\left(x^O\right) : x^O \,\varepsilon\, x\}$, so we start with $f(0) = \{\,\}$, $f(-1) = f(1) = \{\{\,\}\}$ and $f(2) = \{\,\{\},\{\{\,\}\}\,\}$, etc. An infinite descending chain of options starting at $x$ would imply an infinite chain of membership starting at $f(x)$. ☑

Now consider predicates about numbers. A predicate $P$ whose argument is one number is said to be hereditary if $\left(P\left(x^O\right) \forall x^O\right) \implies P(x)$, in other words if the truth of $P$ for all the options of $x$ guarantees the truth of $P$ for $x$: $x$ inherits the truth of $P$ from its options, which, since they have earlier birthdays, are older and can naturally be thought of as its parents. Now here is the remarkable basis for almost all our proofs:

**The First Induction Theorem.** *A hereditary predicate whose argument is a single number is true for all numbers.*

*Proof.* Suppose there were some $x$ for which a hereditary predicate $P$ did not hold. Then it would have to not hold for some option of $x$, and then for an option of that option, and so on, leading to an infinite descending chain of options. ☑

So this is how proof by surreal induction works: we assume a result for all the options of some $x$, and on that basis prove it for $x$ itself, and we shall thereby have proved it for all numbers. Notice that this sort of induction doesn't need to be started. An hereditary predicate does hold for 0, because it holds, albeit vacuously, for all 0's options.

Actually we need a bit more than that theorem. A predicate $P$ about two numbers is said to be hereditary if $\left(\left(P\left(x^O, y\right) \;\&\; P\left(x, y^O\right) \;\&\; P\left(x^O, y^O\right)\right) \forall x^O, y^O\right) \implies P(x, y)$.

**The Second Induction Theorem.** *A hereditary predicate about two numbers is true for all ordered pairs of numbers.*

*Proof.* We can temporarily define an option of an ordered pair $(x, y)$ to be either an $\left(x^O, y\right)$ or an $\left(x, y^O\right)$ or an $\left(x^O, y^O\right)$. Now suppose there were some $(x, y)$ for which a hereditary predicate $P$ did not hold. Then it would have to not hold for some option of $(x, y)$, and then for an option of that option, and so on, leading to an infinite descending chain of this sort of option, which clearly can't happen because x and y can only have finite descending chains of options of the ordinary sort. ☑

The reader is invited to state and prove the Third and Fourth Induction Theorems, which we shall also need. Now we are ready for

**The Order Theorem.**

*(a) For any number $x$, and any of its options, $x^R \not\leq x \not\leq x^L$, and $x \leq x$ (so $x = x$).*

*(b) For any numbers $x$, $y$ and $z$, if $x \leq y \leq z$, then $x \leq z$.*

*(c) For any number $x$, and any of its options, $x^L < x < x^R$.*

*(d) For any numbers $x$ and $y$, $x \leq y$ or $y \leq x$.*

*Proof.*

(a) We use the first induction theorem as we only have one number to consider.

$$\begin{array}{ccc} x^R & \leq & x \quad ? \\ \{\ldots \mid \ldots\} & & \{x^L \mid x^R\} \end{array}$$

Whatever the options of $x^R$ might be, this is false, because, by the inductive hypothesis, $x^R \leq x^R$ so the right number has a right option that is $\leq$ the left number. So $x^R \not\leq x$, and similarly $x \not\leq x^L$.

$$\begin{array}{ccc} x & \leq & x \quad ? \\ \{x^L \mid x^R\} & & \{x^L \mid x^R\} \end{array}$$

This is true, because, by what we have just shown, the right number has no right option $\leq$ the left number, and the left number has no left option that is $\geq$ the right number.

(b) We use the third induction theorem. Assuming $x \leq y \leq z$,

$$\begin{array}{ccc} x & \leq & z \quad ? \\ \{x^L \mid x^R\} & & \{z^L \mid z^R\} \end{array}$$

If $z^R \leq x$, then $z^R \leq x \leq y$, so by the inductive hypothesis, $z^R \leq y$, which contradict $y \leq z$. Similarly we cannot have $x^L \geq z$, so yes, $x \leq z$

(c) Back to the first induction theorem,

$$\begin{array}{ccc} x^L & \leq & x \quad ? \\ \{\left(x^L\right)^L \mid \left(x^L\right)^R\} & & \{x^L \mid x^R\} \end{array}$$

We cannot have $x^R \leq x^L$, for that contradicts $x$ being a number. We have $\left(x^L\right)^L \leq x^L$ by the inductive hypothesis, so if $x \leq \left(x^L\right)^L$, $x \leq x^L$ by part (b), which contradicts part (a). So $x^L \leq x$, but by part (a), $x^L \not\geq x$, so $x^L < x$. Of course $x < x^R$ in the same way.

(d) Suppose $x \not\leq y$. Then either $x \geq y^R > y$ or $y \leq x^L < x$. In either case $y < x$ by part (b). ☑

Henceforth we shan't necessarily make explicit reference to which of the induction theorems we are using. That was the most extraordinary proof, exactly following the logic of [4], but with a few extra words of explanation. It is the very essence of the surreal numbers, and worth reading several times. On the one hand each step is almost trivial, but there is an intricate interconnection of tiny step after tiny step that boots up a total order from nothing. There are some important consequences. Firstly, we now know that equality is an equivalence relation, and we shall call its equivalence classes equality classes. We need to distinguish equality from identicality. We shall use $\equiv$ to signify identicality, as well as for assigning names to numbers, including when we construct a number by specifying its options. ie $x \equiv y$ only when $x$ and $y$ have identical options. When we use $=$, it is a sign that the two numbers very probably have different options.

There is intuitively appealing characterisation of equality. In order to know whether two numbers are equal, we need first to establish the order relationship between each and the options of the other. Having done that:

**The Equality Theorem.** *Numbers $x$ and $y$ are equal unless one is an option of the other, or one of them has an option that lies strictly between them.*

*Proof.* For equality to fail, $x \leq y$ or $y \leq x$ must fail, so, from the definition of $\leq$, we must have one of $y^R \leq x$, $y \leq x^L$, $x^R \leq y$ or $x \leq y^L$. If one of these inequalities is an equality, then one of $x$ and $y$ is an option of the other, and otherwise one of them has an option that lies strictly between them. ☑

More generally, it is the interval between the options that conjures up the new number, rather than the options themselves. For example:

**The Extra Option Theorem.** *If $x \equiv \{x^L \mid x^R\}$ is a number, and $l$ and $r$ are numbers such that $l < x < r$, then*

$$\{l, x^L \mid x^R\} = x = \{x^L \mid r, x^R\}$$

*Proof.* To be clear about the notation, $x^L$ here represents a typical option, of which there may be many. (All the numbers we have met so far are equal to a number with at most two options, but some numbers are not equal to any number with only finitely many options.) Let $y = \{l, x^L \mid x^R\}$. Firstly, this is a number, since $l < x < x^R$.

$$\underset{\{x^L \mid x^R\}}{x} \quad \leq \quad \underset{\{l, x^L \mid x^R\}}{y} \quad ?$$

This is trivially true as the right number's right options are also right options of the left number and therefore $>$ it. Similarly the left number's left options are left options of the right number.

$$\underset{\{l, x^L \mid x^R\}}{y} \quad \leq \quad \underset{\{x^L \mid x^R\}}{x} \quad ?$$

This is again true. The right number's right options work as before, and the left number's left options are either options of $x$ and therefore $<$ it, or are $<$ it by construction.

The case when we add a right option is essentially the same. ☑

Now, we have only met a few numbers so far, but we clearly at the start of a very long journey. In fact there is a surprising consequence on the last two results.

**The Proper Class Theorem.** *The class,* **No***, of all numbers is not a set. For any number $x$ the equality class of $x$ is not a set.*

*Proof.* If **No** were a set, $\{\mathbf{No} \mid \}$ would be a number, and greater than any of its left options, and therefore could not be one of them.

If the class of numbers equal to $x$ were a set, then the class of numbers not equal to $x$ would not be. $x$ has only a set of options, so the class of numbers that are not equal to $x$ and are not options of $x$ is a proper class. They can be added one at a time to the options of $x$ (left or right according as they are less than or greater than $x$), to create a proper class of distinct numbers, that by the extra option theorem are all equal to $x$. ☑

Returning to the issue of having different numbers that are equal. Conway says "... we must distinguish between the *form* $\{L \mid R\}$ of a number and the number itself", but never says what "the number itself" actually is. By the Proper Class Theorem, we cannot define it to be the equivalence class to which the equal forms belong, because that is a proper class, and so we could not have set of numbers. We cannot do it, but at the time of writing that is what the Wikipedia article does. We shall meet another approach later when we shall be able to define a distinguished member of each equality class - a canonical form for the number if you wish. This is much more like the way we handle fractions, $1/2$ being the canonical form for the equality class that also includes $3/6$ and $(-4)/(-8)$. For the moment we just have to live with having different numbers that are equal. Rather than ever talking about "the number itself", we shall say that a property of a number is a "property of the number itself" if it holds for all the numbers in a equality class or none of them. So for example being positive is a property of the number itself, whereas having only one option is not; it is a property of the form of a number.

## 3.3 Surreal Ordinals

Now that we have sorted out the order of the surreals, we are in a position to show that they contain a copy of the ordinals.

**The Definition of Surreal Ordinals.** *A surreal ordinal is a surreal number $\alpha$ such that*

*(a) $\alpha$ has no right options;*

*(b) all the left options of $\alpha$ are surreal ordinals;*

*(c) $\delta \,\varepsilon\, \gamma \,\varepsilon\, \alpha \implies \delta \,\varepsilon\, \alpha$*

This obviously mimics von Neumann's construction, and for the moment we shall call the class of surreal ordinals $\mathbf{On_s}$. Using a subscript $n$ to denote von Neumann ordinals, and $s$ to denote surreal ones, the first few ordinals are:

$$
\begin{array}{rcl}
\text{von Neumann Ordinals} & & \text{surreal ordinals} \\
\{\,\} \equiv 0_n & \sim & 0_s \equiv \{\,\mid\,\} \\
\{0_n\} \equiv 1_n & \sim & 1_s \equiv \{0_s \mid\,\} \\
\{0_n, 1_n\} \equiv 2_n & \sim & 2_s \equiv \{0_s, 1_s \mid\,\} \\
\{0_n, 1_n, 2_n\} \equiv 3_n & \sim & 3_s \equiv \{0_s, 1_s, 2_s \mid\,\} \\
& \vdots & \\
\{0_n, 1_n, 2_n, 3_n, \ldots\} \equiv \omega_n & \sim & \omega_s \equiv \{0_s, 1_s, 2_s, 3_s, \ldots \mid\,\}
\end{array}
$$

 

We can exhibit an explicit ordermorphism. Define $\phi$ on **On** by

$$\phi(\alpha) \equiv \{\phi(\gamma) : \gamma \in \alpha \mid \}$$

**The Ordinal Equivalence Theorem.** *For $\alpha, \gamma \in$ **On**,*

*(a) $\phi(\alpha)$ is a number;*

*(b) $\phi(\alpha)$ is a surreal ordinal;*

*(c) $\gamma < \alpha \implies \phi(\gamma) < \phi(\alpha)$*

*(d) $\delta_s \in \mathbf{On_s} \implies \exists \delta \in \mathbf{On}$ such that $\phi(\delta) = \delta_s$.*

*Proof.*

(a) By induction, all the $\phi(\gamma)$ are numbers, and they form a set (not a proper class) and so, with no right options, $\phi(\alpha)$ is a number.

(b) By induction, all the $\phi(\gamma)$ are ordinals, for $\gamma \in \alpha$. Now suppose $\delta_s$ is a surreal ordinal with $\delta_s \,\varepsilon\, \phi(\gamma)$. Then $\delta_s = \phi(\delta)$ for some $\delta \in \gamma$. This means $\delta \in \alpha$, so $\delta_s \,\varepsilon\, \phi(\alpha)$. So $\phi(\alpha)$ is a surreal ordinal.

(c) $\gamma < \alpha \iff \gamma \in \alpha \iff \phi(\gamma) \,\varepsilon\, \phi(\alpha) \iff \phi(\gamma) < \phi(\alpha)$

(d) All the options of $\delta_s$ are, by induction, of the form $\phi(\gamma)$,where $\gamma \in$ **On**. Then $\phi(\cup_\gamma(\gamma + 1))$ has the same options as $\delta_s$ and is therefore identical to it. ☑

Now we can dispense with the von Neumann ordinals. From now on the ordinals *are* the surreal ordinals, and the class of them all will be **On**. Surreal ordinals are the members of their equality classes that have names, if any do. Eg $3 \equiv \{0, 1, 2 \mid \}$.

## 3.4 Birthdays

This is the proper definition:

**The Definition of Birthday.** *The birthday, $\beta(x)$, of a number $x$ is the smallest ordinal greater than the birthdays of all its options.*

The existence of such an ordinal follows from the Ordinal Theorem - the options form a set, so their birthdays do, so there is an ordinal bigger than them all, so there is a least such ordinal. That is to say $\beta(x) = \sup\left(\beta\left(x^O\right) + 1\right)$. This definition agrees with the informal one we used in constructing our first few number: 0 has birthday 0, 1 and $-1$ have birthday 1, and the 17 new numbers we said were born on day two do indeed have birthday 2, so that equal numbers can have unequal birthdays. There is an important counterpoint to the proper class theorem:

**The Birthday Set Theorem.** *The class $N_\alpha$ of numbers born on any day $\alpha$ form a set.*

*Proof.* By induction the numbers born on each earlier day form a set, so all the numbers born before $\alpha$ form a set, so the class of all subsets of this set is a set, so the class of ordered pairs of such subsets is a set, so the subset of that consisting of numbers is a set. ☑

$N_\alpha$ are the new numbers on day $\alpha$, and we shall also use $O_\alpha = \cup_{\gamma<\alpha} N_\gamma$, the old numbers, and $M_\alpha = \cup_{\gamma\le\alpha} N_\gamma$, the made numbers. Note that $M_\alpha = O_{\alpha+1}$.

A key, but very simple, result is

**The Ordinal's Birthday Theorem.** *An ordinal is its own birthday.*

*Proof.*
$$\begin{aligned}\beta(\alpha) &= \sup\left(\beta\left(\alpha^O\right)+1\right)\\ &= \sup\left(\alpha^O+1\right), \text{ by induction}\\ &= \alpha\end{aligned}$$
☑

The birthday concept gives rise to perhaps the most important way to establish equality. Following Conway, we say that one number is simpler that another if it has an earlier birthday.

**The Simplicity Theorem.** *$x$ is a number, and $z$ is a number with the earliest possible birthday that lies strictly between the left and right options of $x$, then $x = z$.*

*Proof.* We appeal to the equality theorem. The birthday of $z$ is no more than the birthday of $x$, so $x$ cannot be an option on $z$. $z$ cannot be an option on $x$ as it lies strictly between the left and right options of $x$. For the same reason $x$ cannot have an option that lies strictly between $x$ and $z$. However, if $z$ were to have an option that lies strictly between $x$ and $z$, it would be a simpler number than $z$ lying strictly between the left and right options of $x$. So none of the ways in which $x = z$ can fail happen. ☑

There can, up to equality, be only one number between the left and right options of $x$ with the earliest possible birthday, for if there were two, there would be a simpler number lying between them. So we can rephrase the simplicity theorem as saying that $x$ is equal to the simplest number between its options.

It is not necessarily the case that $x$ is equal to a number that lies between its left and right options and is merely simpler than it (rather than simplest). For example 1/8 and 7/8 have birthday 4, so $x \equiv \{1/8 \mid 7/8\}$ has birthday 5. The simplest number between 1/8 and 7/8 is 1/2 which has birthday 2, and indeed $x = 1/2$. 1/4 and 3/4 with birthday 3, and 3/8 and 5/8 with birthday 4 are simpler than $x$, and lie between its left and right options, but are not equal to it.

There is now a corollary to the equality theorem that is frequently useful: two numbers with the same birthday are equal unless there is an earlier number strictly between them.

We shall find it useful to have a parallel definition to that of birthday that gives the same result for all members of an equality class.

**The Definition of Conception Day.** *The conception day $\kappa(x)$ of a number $x$ is the smallest ordinal that is the birthday of a number equal to $x$.*

So for example, whilst the birthday of $\{-1 \mid 1\}$ is 2, its conception day is 0.

# 4 THE BINARY TREE

## 4.1 Ancestors

We remarked earlier that the numbers form a giant binary tree. In this section we make that idea more precise and more useful. Firstly, going forwards in time, it is clear that any number $x \in N_\alpha$ has two immediate descendants in $N_{\alpha+1}$.

$$l(x) = \{y \in M_\alpha : y < x \mid y \in M_\alpha : y \geq x\} \text{ and } r(x) = \{y \in M_\alpha : y \leq x \mid y \in M_\alpha : y > x\}$$

Clearly $l(x) < x < r(x)$, and there is no number $z$ in $M_{\alpha+1}$ with $l(x) < z < x$ or $x < z < r(x)$. Now let's look at what happens when we go backwards in time.

**The Definition of Ancestors.** *Suppose $x$ is a number whose conception day is $\alpha$. Then for $\gamma \leq \alpha$, define $x_\gamma$, the $\gamma$-ancestor of $x$, by*

$$x_\gamma \equiv \{y \in O_\gamma : y < x \mid y \in O_\gamma : y > x\}$$

So $x_\gamma$ is the nearest you can get to $x$ on day $\gamma$. Note that for $\gamma > \kappa(x)$, $x_\gamma$ is undefined, even if $\gamma \leq \beta(x)$. An ancestor of $x$ is called a left ancestor if it is less than $x$ and a right ancestor if it is greater than $x$. The $\gamma$-ancestor of $x$ is called a proper ancestor if $\gamma < \alpha$. The sequence of the ancestors of $x$ is called the ancestry of $x$. Since the ancestry of a number depends only upon its conception day and its order relationship to other numbers, equal numbers have the same ancestry: ancestry is a property of the number itself.

**The Ancestor Theorem.** *For $x$ with $\kappa(x) = \alpha$, and $\gamma \leq \alpha$:*

*(a) $x_\gamma \in N_\gamma$, and $x_\gamma \neq x$ for $\gamma < \alpha$.*

*(b) $x_0 \equiv 0$ and $x_\alpha = x$.*

*(c) If $\delta < \gamma$, $(x_\gamma)_\delta \equiv x_\delta$.*

*(d) If $x_\gamma < x$ then $x_{\gamma+1} \equiv r(x_\gamma)$, whilst if $x_\gamma > x$ then $x_{\gamma+1} \equiv l(x_\gamma)$.*

*(e) Any two numbers $x$ and $y$ have a latest (ie youngest) common ancestor, which lies between them or is equal to one of them.*

*(f) $x = \tilde{x} \equiv \{x_\gamma : \gamma < \alpha, x_\gamma < x \mid x_\gamma : \gamma < \alpha, x_\gamma < x\}$. Ie $x$ is equal to the number formed by treating the proper ancestors of $x$ as options, left ancestors becoming left options, and right ancestors right oprions.*

*Proof.*

(a) $x_\gamma$ is clearly a number, and a member of $M_\gamma$. $x_\gamma$ cannot be in $O_\gamma$, else it would be one of its own options. So $x_\gamma \in N_\gamma$. If $\gamma < \alpha$, $x \notin N_\gamma$, so $x_\gamma \neq x$.

(b) $O_0 \equiv \{\}$, so $x_0 \equiv \{\,|\,\} \equiv 0$. $x$ is equal to the simplest number between its left and right options, and it lies between the left and right options of $x_\alpha$, which include the left and right options if $x$, so a fortiori it is equal to the simplest number between them, which is $x_\alpha$.

(c)
$$x_\delta \equiv \{y \in O_\delta : y < x \mid y \in O_\delta : y > x\}$$
$$(x_\gamma)_\delta \equiv \{y \in O_\delta : y < x_\gamma \mid y \in O_\delta : y > x_\gamma\}$$
but nothing in $O_\delta$, or indeed the larger $O_\gamma$, can lie between $x_\gamma$ and $x$.

(d) If $x_\gamma < x$, then $\gamma < \alpha$, and

$$x_{\gamma+1} \equiv \{y \in M_\gamma : y < x \mid y \in M_\gamma : y > x\}$$
$$r(x_\gamma) \equiv \{y \in M_\gamma : y \leq x_\gamma \mid y \in M_\gamma : y > x_\gamma\}$$

Because $x$ itself in not in $M_\gamma$ these can only fail to be equal in there some $y \in M_\gamma$ with $x_\gamma < y < x$. If $y \in O_\gamma$, it would, being less than $x$, be a left option of $x_\gamma$, which contradicts $x_\gamma < y$, whilst if $y \in N_\gamma$, there would be some $z \in O_\gamma$ with $x_\gamma < z < y$, which would similarly be a left option of $x_\gamma$. So $x_{\gamma+1} \equiv r(x_\gamma)$. Similarly if $x_\gamma > x$ then $x_{\gamma+1} \equiv l(x_\gamma)$.

 

(e) Certainly there is an ordinal $\gamma$ which is the smallest for which either $x_\gamma \neq y_\gamma$ or at least one of $x_\gamma$ and $y_\gamma$ is undefined.

$\gamma > 0$, because 0 is an ancestor of all numbers. Now we show that $\gamma$ is not a limit ordinal. If at least one of $x_\gamma$ and $y_\gamma$ is undefined, then, assuming without loss of generality that $\kappa(x) \geq \kappa(y)$, $\gamma = \kappa(y) + 1$, which is not an limit ordinal. Otherwise suppose without loss of generality that $x_\gamma < y_\gamma$. There is some $z$ with $\beta(z) = \delta < \gamma$, and $x_\gamma < z < y_\gamma$. Then $x_{\delta+1} = (x_\gamma)_{\delta+1} < z < (y_\gamma)_{\delta+1} = y_{\delta+1}$, which contradicts the definition of $\gamma$ if it is a limit ordinal.

So now $\delta = \gamma - 1$ is the largest ordinal for which $x_\delta = y_\delta$. If $x_{\delta+1}$ is undefined, then $x_\delta = x$, and similarly if $y_{\delta+1}$ is undefined. Otherwise, by part (d), $x_{\delta+1}$ and $y_{\delta+1}$ must both be equal to one of $l(x_\delta)$ and $r(x_\delta)$, but they cannot both be equal to the same one, so $x_\delta$ must be a left option of one of $x$ and $y$, and a right option of the other, and therefore lies between them.

(f) Suppose otherwise, and assume without loss of generality that $x < \tilde{x}$. They have a common ancestor $x_\gamma$ with $\gamma < \alpha$ and $x < x_\gamma < \tilde{x}$. $x_\gamma$ is a right ancestor of $x$ and therefore a right option of $\tilde{x}$, which is a contradiction. ☑

From part (f) we see that if we know all the ancestors of $x$ before some ordinal $\gamma$, and whether they are left or right ancestors, that is enough information to construct the $\gamma$-ancestor, without direct reference to $x$. Applying that concept recursively tells us that just knowing whether each ancestor of $x$ is a left ancestor or a right ancestor is enough to recover $x$, which leads us onto the next concept.

## 4.2 The Sign Expansion

**The Definition of the Sign Expansion.** *For number a $x$ with $\kappa(x) = \alpha$, the sign expansion of $x$ is a sequence of signs ($+$ or $-$) of order type $\alpha$, where, for $\gamma < \alpha$, the $\gamma^{th}$ sign, $\sigma_\gamma(x)$, is $+$ or $-$ according as $x_\gamma$ is a left or right ancestor.*

It in helpful to visualize the sign expansion interleaving the list of ancestors,

$$\begin{aligned} 3 &: 0 + 1 + 2 + 3 \\ -2 &: 0 - (-1) - (-2) \\ 3/2 &: 0 + 1 + 2 - 3/2 \\ \omega &: 0 + 1 + 2 + 3 \cdots \omega \end{aligned}$$

Of course the left ancestors are precisely the ones followed by a $+$, and the right ones by a $-$. We shall see more examples later.

**The Sign Expansion Theorem.** *For numbers $x$ and $y$, their sign expansions are equal iff $x = y$, and the sign expansion of $x$ is lexicographically less than that of $y$ iff $x < y$.*

*Proof.* Suppose that for some $\gamma$ the sign expansion of $x$ is equal to that of $y$ for $\delta < \gamma$, but then, say, the $\sigma_\gamma(x) = -$ whilst for $\sigma_\gamma(y) = +$. Then $x_\gamma \equiv y_\gamma$, but $x_\gamma$ is right ancestor for $x$ but a left ancestor for $y$, so $x < x_\gamma < y$. If no such $\gamma$ exists, and $x$ and $y$ have the same birthday, then they are equal. Otherwise, suppose $y$ had has a longer sign expansion. Then $x$ is an ancestor of $y$, and $y$ has a $+$ or $-$ on $x$'s birthday according as it is greater or less than $x$. ☑

 

Some authors, notably Gonshor in [8], use the sign expansion as the definition of a number. That is to say that they define a number to be a map from an ordinal to $\{+, -\}$. This has two advantages. Firstly it is a less mysterious start. Secondly we don't have distinct numbers being equal. Conway objects to this approach on two grounds. Firstly, one needs to have defined the ordinals in advance, whereas they emerge naturally as a part of **No** if we start from double sets, and in this paper they emerge at the earliest possible moment: as soon as the properties of $\leq$ are established. Secondly, it misses the wonderful sense of creating something out of nothing that comes from his amazing definition. It seems to me that a third reason is that whilst it is easy to create the sign expansion from the double set definition of numbers, that other way round is really hard work - Gonshor takes over two closely typed pages with four cases, of which two have two sub-cases, to prove that given two sets $L$ and $R$ of sign expansions with $L < R$ there is a unique sign expansion of shortest length between them. For a fourth reason, addition and multiplication cannot be defined from the sign expansion directly, but only from the Conway form. That seems to me to be telling us that the proper place to start is with double sets.

There is a fascinating corollary to the Ancestor Theorem. Recall that part (f) that a number $x$ is defined, up to equality, by treating is ancestors as its options:

$$x = \tilde{x} \equiv \{x_\gamma : \gamma < \alpha, x_\gamma < x \mid x_\gamma : \gamma < \alpha, x_\gamma < x\}$$

We can have a recursive version of that:

$$x = \hat{x} \equiv \{\hat{x_\gamma} : \gamma < \alpha, x_\gamma < x \mid \hat{x_\gamma} : \gamma < \alpha, x_\gamma < x\}$$

From what we have seen, it is obvious that each number is equal to precisely one of these hatted numbers, so they are distinguished members of each equality class, which I shall call Conway-Gonshor numbers. They can be thought of as canonical versions of numbers, even as the "numbers themselves". Some of the first few are:

$$\begin{aligned}
\hat{0} &\equiv \{\,\mid\,\} \equiv 0 \\
\hat{1} &\equiv \{\hat{0} \mid\,\} \equiv \{0 \mid\,\} \equiv 1 \\
\hat{2} &\equiv \{\hat{0}, \hat{1} \mid\,\} \equiv \{0, 1 \mid\,\} \equiv 2 \\
\widehat{\{2 \mid\,\}} &\equiv \{\hat{0}, \hat{1}, \hat{2} \mid\,\} \equiv \{0, 1, 2 \mid\,\} \equiv 3 \\
\widehat{1/2} &\equiv \{\hat{0} \mid \hat{1}\} \equiv \{0 \mid 1\} \equiv 1/2
\end{aligned}$$

The final identity in each line is no coincidence, because it is the Conway-Gonshor numbers in each equality class that I have chosen to give the special name to. Delightfully, the numbers in the ordinal equality classes that were earlier picked out for naming because their options mimic the von Neumann construction (choosing to give the name "3" to $\{0, 1, 2 \mid\,\}$ rather than, for example, $\{2 \mid\,\}$) are in fact the Conway-Gonshor numbers, because the ancestors of an ordinal are all the smaller ordinals. For Gonshor, an ancestor of a number is just a truncation of it (as a sign expansion of course), and he constructs these numbers (his theorem 2.8) as canonical versions of Conway's construction, so they are where the two approaches meet in the middle. We shall call the class of Conway-Gonshor numbers **CG**.

# 5 ARITHMETIC

It is time to justify the word "number" by defining addition and multiplication.

## 5.1 Addition

The definition is of course going to be recursive, and we know that if addition is going to behave as we would like, then $x + y$ must be greater than $x + y^L$, and so on. Asserting thes relationships like this are to be true turns out to be enough to define addition completely.

**The Definition of Addition.**

$$x + y \equiv \{x^L + y, x + y^L \mid x^R + y, x + y^R\}$$

To be clear what this means, every option $x^O$ of $x$ creates an option $x^O + y$ of $x + y$, as does every option of $y$, and every option of $x + y$ arises in one of these two ways. So we can write this slightly more succinctly as

$$(x + y)^O \equiv x^O + y \vee x + y^O$$

Here we are treating $O$ as a variable that can take the values $L$ and $R$, and this version of the definition means that any option of $x$ or $y$ leads to an option of $x + y$ according to the rule given, and all options of $x + y$ arise in this way. Now, we can imagine starting at the beginning, finding 0+0, then 0+1, 1+0 and so on, but we can speed things up by starting with a general theorem.

**The Addition Theorem.** *For all numbers $x$, $y$ and $z$, we have*

*(a)* $x + 0 \equiv x$;

*(b)* $x + y \equiv y + x$;

*(c)* $x + (y + z) \equiv (x + y) + z$;

*(d)* $x \leq y$ *iff* $x + z \leq y + z$;

*(e)* $x + y$ *is a number*

*Proof.* We treat the whole theorem as a single predicate, proving it by induction, using the third induction theorem as there are three variables, so that we can assume that expressions like $x + y^O$ are numbers.

(a)
$$\begin{aligned} (x + 0)^O &\equiv x^O + 0, \text{ as 0 has no options} \\ &\equiv x^O, \text{ by induction} \end{aligned}$$

(b)
$$\begin{aligned} (x + y)^O &\equiv x^O + y \vee x + y^O \\ &\equiv y + x^O \vee y^O + x, \text{ by induction} \\ &\equiv (y + x)^O \end{aligned}$$

(c)
$$\begin{aligned} (x + (y + z))^O &\equiv x^O + (y + z) \vee x + (y + z)^O \\ &\equiv x^O + (y + z) \vee x + (y^O + z) \vee x + (y + z^O) \\ &\equiv (x^O + y) + z \vee (x + y^O) + z \vee (x + y) + z^O, \text{ by induction} \\ &\equiv (x + y)^O + z \vee (x + y) + z^O \\ &\equiv ((x + y) + z)^O \end{aligned}$$

(d) Suppose $x + z \leq y + z$:

$$\underset{\{x^L \mid x^R\}}{x} \quad \leq \quad \underset{\{y^L \mid y^R\}}{y} \quad ?$$

This is true, because $y^R \leq x \implies y^R + z \leq x + z$ by induction, so that $y^R + z \leq y + z$ which is not possible, because $y^R + z$ is a right option of $y + z$, and $x^L \geq y \implies x^L + z \geq y + z \geq x + z$ which is also not possible, because $x^L + z$ is a left option of $x + z$.

Now the other way round, suppose $x \leq y$:

$$\underset{\{x^L + z, x + z^L \mid x^R + z, x + z^R\}}{x + z} \quad \leq \quad \underset{\{y^L + z, y + z^L \mid y^R + z, y + z^R\}}{y + z} \quad ?$$

$y^R + z \leq x + z \implies y^R \leq x$, by what we have just shown, which is not possible. $y + z^R \leq x + z \implies y + z^R < x + z^R \implies y < x$, which is not true. The left options for falsifying $x + z \leq y + z$ fail in exactly the same way, so it is true. We deduce that also $x = y$ iff $x + z = y + z$, and $x < y$ iff $x + z < y + z$.

(e) By induction all the options of $x + y$ are numbers, and

$$x^L + y, x + y^L < x + y < x^R + y, x + y^R$$

so $x + y$ is a number. ☑

So now at last we know that the name we gave to $\{\,|\,\}$ was the correct one. We can't show that for $\{0 \mid\}$ until we have defined multiplication, but if we accept that for the moment, we can verify some of the others.

$$\begin{array}{llll}
 & \underset{\{0 \mid\}}{1} \quad + \quad \underset{\{0 \mid\}}{1} & & \\
\equiv & \{0 + 1, 1 + 0 \mid\} & & \\
\equiv & \{1 \mid\} & & \\
= & \{0, 1 \mid\} & & \text{, by the extra option theorem} \\
\equiv & 2 & &
\end{array}$$

This incidentally illustrates that the sum of two Conway-Gonshor numbers is not necessarily a Conway-Gonshor number.

$$\begin{array}{ll}
 & \underset{\{0 \mid\}}{1} \quad + \quad \underset{\{0 \mid 1\}}{1/2} \\
\equiv & \{0 + 1/2, 1 + 0 \mid 1 + 1\} \\
\equiv & \{1/2, 1 \mid 2\} \\
= & \{0, 1 \mid 2\} \\
\equiv & 3/2
\end{array}$$

One more example:

$$\begin{array}{ll}
 & \underset{\{0 \mid 1\}}{1/2} \quad + \quad \underset{\{0 \mid 1\}}{1/2} \\
\equiv & \{0 + 1/2, 1/2 + 0 \mid 1 + 1/2, 1/2 + 1\} \\
\equiv & \{1/2 \mid 3/2\} \\
= & 1
\end{array}$$

The reader is invited to verify a few more names such as 3 and 1/4.

The numbers have an obvious symmetry between left and right, so the definition of $-x$ should come as no surprise, and the theorem about it is straightforward as well. Option superscripts have higher precedence than unary minus, so $-x^O$ means $-(x^O)$.

**The Definition of the Negative.**

$$-x \equiv \{-x^R \mid -x^L\}$$

We can make the symbols $L$ and $R$ into a group, which of course is $Z_2$, as follows:

$$L + L = R + R = R, \qquad L + R = R + L = L,$$

This little group doesn't gain us much here; it really comes into its own later with the multiplicative operations. Using it here anyway, we can restate the definition of $-x$ as

$$(-x)^{L+O} = -x^O$$

This means that every option of $x$ gives rise to an option if $-x$ according to this rule, and every option of $-x$ arises in this way.

**The Negative Theorem.** *For all numbers $x$ and $y$, we have*

*(a)* $-(-x) \equiv x$*;*

*(b)* $-(x + y) \equiv -x + -y$*;*

*(c)* $x \leq 0$ *iff* $-x \geq 0$*;*

*(d)* $x + (-x) = 0$*.*

*(e)* $-x$ *is a number;*

*Proof.*

(a)
$$\begin{aligned}(-(-x))^{L+O} &\equiv -(-x)^O \\ &\equiv -(-x^{L+O}) \\ &\equiv x^{L+O}, \text{ by induction}\end{aligned}$$

(b)
$$\begin{aligned}(-(x + y))^{L+O} &\equiv -(x + y)^O \\ &\equiv -(x^O + y) \vee -(x + y^O) \\ &\equiv -x^O + -y \vee -x + -y^O, \text{ by induction} \\ &\equiv (-x)^{L+O} + -y \vee -x + (-y)^{L+O} \\ &\equiv (-x + -y)^{L+O}\end{aligned}$$

(c)
$$\begin{aligned}x \leq 0 &\iff \nexists\, x^L \geq 0 \\ &\iff \nexists\, x^L \text{ such that } -x^L \leq 0, \text{ by induction} \\ &\iff \nexists\, (-x)^R \leq 0 \\ &\iff -x \geq 0\end{aligned}$$

(d) $$\begin{aligned} x+(-x) &\equiv \{x^L + -x, x+(-x)^L \mid x^R + -x, x+(-x)^R\} \\ &\equiv \{x^L + -x, x + -x^R \mid x^R + -x, x + -x^L\} \end{aligned}$$

$x < x^R$, so $x + -x^R < x^R + -x^R = 0$ by induction, and $x > x^L$, so $x + -x^L > x^L + -x^L = 0$, by induction, and therefore $0 > -(x + -x^L) = x^L - x$. The right options are similar. ☑

This implies that the sum or difference of equal numbers are equal, so that the sum of two numbers is a property of the numbers themselves. **No** is a totally ordered Abelian group under addition, or if you are squeamish about a group having so many equal elements, you can say **CG** is a totally ordered Abelian group under the operation $\oplus$ defined by

$$\hat{x} \oplus \hat{y} \equiv \hat{z}, \text{ where } z \equiv \hat{x} + \hat{y}$$

We can now see that all the negative names we gave to numbers earlier were the correct names. We also need to confirm that our use of $\alpha + 1$ to mean the successor of an ordinal is correct:

$$\begin{aligned} \alpha + 1 &= \{\gamma : \gamma < \alpha \mid\} + \{0 \mid\} \\ &= \{\gamma + 1 : \gamma < \alpha, \alpha \mid\} \\ &= \text{the successor of } \alpha \end{aligned}$$

Otherwise this addition applied to the ordinals is not the same as standard ordinal arithmetic, after all it is commutative. (It is in fact the same as what is called the maximal or natural sum. The natural sum of two ordinals is the largest order type that can be made by interleaving sets of order types corresponding to the two ordinals.)

## 5.2 Multiplication

Multiplication is much more complicated than addition. To begin with it is not at all clear what $xy$ is definitely bigger or smaller than, to make putative options. However $x > x^L$ and $y > y^L$, so $(x - x^L)(y - y^L) > 0$, meaning that $xy > xy^L + x^Ly - x^Ly^L$, so $x^Ly + xy^L - x^Ly^L$ is a potential left option, as, in a similar way, is $x^Ry + xy^R - x^Ry^R$, and there are two similarly derived right options, $x^Ly + xy^R - x^Ly^R$ and $x^Ry + xy^L - x^Ry^L$. Amazingly, once again, those are it all it takes to pin the product down.

**The Definition of Multiplication.**

$$xy \equiv \{x^Ly + xy^L - x^Ly^L, x^Ry + xy^R - x^Ry^R \mid x^Ly + xy^R - x^Ly^R, x^Ry + xy^L - x^ry^L\}$$

It is important to remember that in any one option of $xy$ there is only one option of $x$ and only one of $y$, ie the two occurrences of, say, $x^L$ refer to the same left option of $x$. As with addition, there is a more succinct way to express this:

$$(xy)^{L+O+P} \equiv x^Oy + xy^P - x^Oy^P$$

Here each of $O$ and $P$ can be either $L$ or $R$, and of course each occurrence if $x^O$ represents the same option of $x$, and similarly for $y^P$. The formula means that given an option of $x$ and an option of $y$, an option of $xy$ can be constructed this way, and all options of $xy$ arise in this way.

It is interesting that the definition of multiplication depends upon addition; you can't define multiplication until you've defined addition. The axioms for a field do not make that obvious, although you do at need least to define the 0 element before you can define multiplication because it has a special place in the axioms for multiplication.

 

**The Multiplication Theorem.** *For all numbers $x$, $y$ and $z$*

*(a)* $x0 \equiv 0$;

*(b)* $x1 \equiv x$;

*(c)* $xy \equiv yx$;

*(d)* $(-x)y \equiv x(-y) \equiv -xy$;

*(e)* $(x+y)z = xz + yz$;

*(f)* $(xy)z = x(yz)$;

*(g)* $xy$ *is a number;*

*(h)* $x = y \implies xz = yz$;

*(i)* $x \leq y$ *and* $z \leq t \implies xt + yz \leq xz + yt$. *The third inequality is strict if the first two are.*

*Proof.* As before the theorem is proved as a single predicate, using, since there are four variables, the fourth induction theorem.

(a) 0 has no options, so $x0$ has no options.

(b) 1 has only a left option of 0, so

$$\begin{aligned}(x1)^O &\equiv (x1)^{L+O+L}\\ &\equiv x^O 1 + x1^L - 1 \times 1^L\\ &\equiv x^O 1 + x0 - 1 \times 0\\ &\equiv x^O, \text{ by induction and the previous part}\end{aligned}$$

(c)

$$\begin{aligned}(xy)^{L+O+P} &\equiv x^O y + xy^P - x^O y^P\\ &\equiv y^P x + yx^O - y^P x^O, \text{ by induction}\\ &\equiv (yx)^{L+O+P}\end{aligned}$$

(d)

$$\begin{aligned}((-x)y)^{L+O+P} &\equiv (-x)^O y + (-x)y^P - (-x)^O y^P\\ &\equiv -x^{L+O} y - xy^P + x^{L+O} y^P, \text{ by induction}\\ &\equiv -(xy)^{O+P}\\ &\equiv (-xy)^{L+O+P}\end{aligned}$$

and similarly for $x(-y)$.

(e)

$$\begin{aligned}((x+y)z)^{L+O+P} &\equiv (x+y)^O z + (x+y)z^P - (x+y)^O z^P\\ &\equiv (x^O + y)z + (x+y)z^P - (x^O + y)z^P \vee\\ &\qquad (x + y^O)z + (x+y)z^P - (x+y^O)z^P\\ &= x^O z + yz + xz^P + yz^P - x^O z^P - yz^P \vee\\ &\qquad xz + y^O z + xz^P + yz^P - xz^P - y^O z^P, \text{ by induction}\\ &= (xz)^{L+O+P} + yz \vee xz + (yz)^{L+O+P}\\ &\equiv (xz+yz)^{L+O+P}\end{aligned}$$

Notice that the second equality is not an identity, as $z + (-z) = 0$ has been invoked.

(f) $$\begin{aligned}((xy)z)^{O+P+Q} &\equiv (xy)^{L+O+P}z + (xy)z^Q - (xy)^{L+O+P}z^Q \\ &\equiv (x^Oy + xy^P - x^Oy^P)z + (xy)z^Q - (x^Oy + xy^P - x^Oy^P)z^Q \\ &= (x^Oy)z + (xy^P)z - (x^Oy^P)z + (xy)z^Q - (x^Oy)z^Q - (xy^P)z^Q + (x^Oy^P)z^Q \\ &\qquad\text{, by the above} \\ &= x^O(yz) + x(y^Pz) - x^O(y^Pz) + x(yz^Q) - x^O(yz^Q) - x(y^Pz^Q) + x^O(y^Pz^Q) \\ &\qquad\text{, by induction} \\ &= x^O(yz) + x(y^Pz + yz^Q - y^Pz^Q) - x^O(y^Pz + yz^Q - y^Pz^Q) \\ &\equiv ((x(yz))^{O+P+Q}\end{aligned}$$

(g) For the last three parts of the proof, we extend our notation to include $x^M$ and $x^S$. $x^M$ is just another left option of $x$, maybe the same as $x^L$, maybe different. Similarly $x^S$ is another right option.

For the first of the three, we need to prove four inequalities, of which the first is $x^Ly+xy^L-x^Ly^L < x^My + xy^S - x^My^S$. If $x^L \leq x^M$,

$$(x^L \leq x^M)\&(y^L < y)\text{, so } x^Ly + x^My^L \leq x^Ly^L + x^My\text{, by induction} \tag{1}$$
$$(x^M < x)\&(x^L < y^S)\text{, so } x^My^S + xy^L < x^My^L + xy^S\text{, similarly} \tag{2}$$

Now, all these products are, by induction, numbers, and so can be additively manipulated as members of an ordered Abelian group. Adding the inequalities yields $x^Ly+xy^L-x^Ly^L < x^My + xy^S - x^My^S$. If $x^M \leq x^L$,

$$(x^M \leq x^L)\&(y < y^S)\text{, so } x^Ly + x^My^L \leq x^Ly^L + x^My\text{, by induction} \tag{3}$$
$$(x^M < x)\&(x^L < y^S)\text{, so } x^My^S + x^Ly < x^My + x^Ly^S\text{, similarly} \tag{4}$$

Adding these inequalities again yields $x^Ly + xy^L - x^Ly^L < x^My + xy^S - x^My^S$, as required.

Swapping $x$ and $y$ in the above proof, using the fact we now know that multiplication is commutative, shows that $x^Ly + y^Lx - x^Ly^L < x^Sy + xy^M - x^Sy^M$.

Swapping $L$ with $R$ and $M$ with $S$ reverses the first two inequalities in lines (1), (2), (3) and (4) and therefore preserves the third inequality, proving that $x^Ry + xy^R - x^Ry^R < x^Sy + xy^M - x^Sy^M$.

Finally swapping $L$ with $R$ and $M$ with $S$ and $x$ with $y$ proves that $x^Ry + y^Rx - x^Ry^R < x^My + xy^S - x^My^S$.

(h) If $x = y$ have $x^L < y < x^R$ and $y^L < x < y^R$. Also inductively we know that $xz^O = yz^O$, so using further induction,

$(x^L < y)\&(z^L < z)$, so $x^Lz + yz^L < x^Lz^L + yz$, but $yz^L = xz^L$ by induction, so
$x^Lz + xz^L - x^Lz^L < yz$
$(x^L < y)\&(z < z^R)$, so $x^Lz + yz^R > x^Lz^R + yz$, but $yz^R = xz^R$ by induction, so
$x^Lz + xz^R - x^Lz^R > yz$

Now swapping $L$ and $R$ as we did before will show that $x^Rz + xz^R - x^Rz^R < yz$ and $x^Rz + xz^L - x^Rz^L > yz$, so overall we have $(xz)^L < yz < (xz)^R$. Swapping $x$ and $y$ shows that $(yz)^L < z < (yz)^R$, so that $xz = yz$.

 

(i) If $x = y$, then $xt = yz$ and $xt = yt$, so $xt + yz = xz + yt$, and similarly if $z = t$. So assume $x < y$ and $z < t$. Given $x < y$, there must be either some $x^R$ for which $x < x^R \leq y$ or some $y^L$ such that $x \leq y^L < y$.

$$(x < x^R)\&(z < t) \implies xt + x^R z < xz + x^R t$$
$$(x^R \leq y)\&(z < t) \implies x^R t + yz \leq x^R z + yt$$

Adding these yields $xt + yz < xz + yt$. Similarly

$$(y^L < y)\&(z < t) \implies y^L t + yz < y^L z + yt$$
$$(x \leq y^L)\&(z < t) \implies xt + y^L z \leq xz + y^L t$$

Adding these also yields $xt + yz < xz + yt$. ☑

Now we know that 1 was indeed the correct name to give $\{0 \mid \}$. The other key implication of this theorem is that equal numbers have equal products, so that the product of two numbers is a property of the numbers themselves. Also, although we cannot yet divide a general number by number $x$, it is clear from the above theorem that $xz = yx \implies x = y$, so we can unambiguously divide a multiple of $x$ by $x$.

If we try to define the reciprocal of a positive number, we do at least have some obvious options, and might try, for $x > 0$,

$$1/x \equiv \{1/x^R \mid 1/x^L\}$$

However this does not work, and nothing as simple can possibly work, because whilst 3 is equal to a number with only one option, namely $\{2 \mid \}$, 1/3 must have an infinite birthday, and needs an infinite number of options if they are all to be dyadic rationals. It turns out that the proper definition requires one to consider finite sequences of positive options, possibly repeated of course. Let such a sequence be

$$\mathcal{O} = (x^{O_0}, x^{O_1} \ldots, x^{O_{n-1}})$$

Here $x^{O_i}$ is the $i^{\text{th}}$ option, so that some of the $O$s will be $L$s and some will be $R$s. Of course $O_i = O_j$ does not imply that $x^{O_i} = x^{O_j}$, merely that they are the same sort of option.

**The Definition of the Reciprocal.** *For a positive number $x$, define $1/x$ by*

$$(1/x)^{L + \sum O_i} \equiv \frac{1 - \prod\left(1 - x/x^{O_i}\right)}{x}$$

*as $\mathcal{O}$ ranges over all finite sequences of positive options of $x$.*

As before, this means that every finite sequence of positive options of $x$ defines a option of $1/x$ according to this rule, and all options of $1/x$ arise in this way. There are several things to observe about this definition before we start proving anything:

- A sequence with an even number of left options generates a left option however many right options it has, and one with an odd number of left options generates a right option.
- If we expand out the product, the first term is 1 and all the others are multiples of $x$, so the numerator of the big fraction is a multiple of $x$, so the definition is not circular.

 

- The empty sequence is allowed, and it generates a left option of 0. That is the only option for the unique (up to equality) positive number that has no positive options, namely 1, and so we get the right answer for 1/1.
- A sequence of length one generates the option $(1/x)^{L+O} \equiv 1/x^O$, so this construct builds on our first guess.
- If $x \equiv \{2 \mid \}$, our sequences are all just sequences of 2's, generating alternately left and right options of 0, 1/2, 1/4, 3/8, 5/16 etc, which looks promising.

The definition is complicated, but the proof that it works is surprisingly simple.

**The Reciprocal Theorem.** *For any positive number $x$,*

*(a)* $1/x$ *is a number.*

*(b)* $x(1/x) = 1$.

*Proof.*

(a) $1 - x/x^{O_i}$ is never 0, and is negative iff $O_i = L$. So $\prod\left(1 - x/x^{O_i}\right)$ is not 0, and is $< 0$ iff $\sum O_i = L$. So $x(1/x)^L < 1 < x(1/x)^R$, so $(1/x)^L < (1/x)^R$. So $1/x$ is a number.

(b) An option of $x(1/x)$ is constructed from an option of $x$ and an option of $1/x$, so

$$\begin{aligned}
(x(1/x))^{O+\sum O_i} &\equiv x^O(1/x) + x(1/x)^{L+\sum O_i} - x^O(1/x)^{L+\sum O_i} \\
&\equiv x^O(1/x) + x\left(\frac{1-\prod\left(1-x/x^{O_i}\right)}{x}\right) - x^O\left(\frac{1-\prod\left(1-x/x^{O_i}\right)}{x}\right) \\
&= 1 + x^O\left(1/x - \frac{\prod\left(1-x/x^{O_i}\right)}{x^O} - \frac{1-\prod\left(1-x/x^{O_i}\right)}{x}\right) \\
&= 1 + x^O\left(1/x - \left(\frac{1-\left(1-x/x^{O}\right)\prod\left(1-x/x^{O_i}\right)}{x}\right)\right) \\
&\equiv 1 + x^O\left(1/x - (1/x)^{L+O+\sum O_i}\right)
\end{aligned}$$

So a left option of $x(1/x)$ is less than 1, whilst a right option is greater then 1. $x(1/x)$ is certainly greater than 0, so by the simplicity theorem, $x(1/x) = 1$. ☑

To summarise the whole section, **No** is a totally ordered field, or if you prefer, **CG** is a totally ordered field under the operations

$$\hat{x} \oplus \hat{y} \equiv \hat{z}, \text{ where } z \equiv \hat{x} + \hat{y}$$
$$\hat{x} \otimes \hat{y} \equiv \hat{z}, \text{ where } z \equiv \hat{x}\hat{y}$$

Here is a simple theorem we shall need later

**The Birthday Addition Theorem.** *For numbers $x$ and $y$,*

*(a)* $\beta(x+y) \le \beta(x) + \beta(y)$

*(b)* $\kappa(x+y) \le \kappa(x) + \kappa(y)$

*Proof.*

(a)
$$\begin{aligned}\beta(x+y) &= \sup \beta(x+y^O)+1, \beta(x^O+y)+1 \\ &\le \sup \beta(x)+\beta(y^O)+1, \beta(x^O)+\beta(y)+1, \text{ by induction} \\ &= \beta(x)+\beta(y)\end{aligned}$$

(b)
$$\begin{aligned}\kappa(x+y) &= \kappa(\hat{x}+\hat{y}) \\ &\le \beta(\hat{x}+\hat{y}) \\ &\le \beta(\hat{x})+\beta(\hat{y}) \\ &= \kappa(\hat{x})+\kappa(\hat{y}) \\ &= \kappa(x)+\kappa(y)\end{aligned}$$

☑

It is an interesting question whether or not $\beta(xy) \le \beta(x)\beta(y)$. It is an obvious enough conjecture, with no obvious counter-examples, but so far as I an aware, no proof either. Gonshor in [8] proves the much weaker result that $\beta(xy) \le 3^{\beta(x)+\beta(y)}$, whilst in [7] the authors prove a significant special case of the obvious conjecture.

# 6 MORE NUMBERS

Now we are in a position to generate and name a whole surreal zoo of numbers, but we'll start with the relatively mundane real numbers.

## 6.1 The Surreal Real Numbers

Earlier we found the ordinals in **No**. The finite ones are also the first step in finding the real numbers in **No**, but for that to work, we need to show that arithmetic works on them as expected, so that the embedding is a ring isomorphism. We prove this by induction, but it is finite induction, rather than transfinite induction, and so it needs starting. We have in fact already done that by showing that $x+0=x$ and $x0=0$. Let $\phi$ be the embedding of the natural numbers in the surreal numbers as surreal ordinals. Then, for $n>0$ and $m>0$:

$$\begin{aligned}\phi(n)+\phi(m) &= \{\phi(n-1) \mid \} + \{\phi(m-1) \mid \} \\ &= \{\phi(n)+\phi(m-1), \phi(n-1)+\phi(m) \mid \} \\ &= \{\phi(n+m-1) \mid \}, \text{ by induction} \\ &= \phi(n+m) \\ \phi(n)\phi(m) &= \{\phi(n-1) \mid \}\{\phi(m-1) \mid \} \\ &= \{\phi(n-1)\phi(m)+\phi(n)\phi(m-1)-\phi(n-1)\phi(m-1) \mid \} \\ &= \{\phi(nm-1) \mid \}, \text{ by induction and the previous part} \\ &= \phi(nm)\end{aligned}$$

From now on these surreal numbers *are* the natural numbers, $\mathbb{N}$, so we have no further need of $\phi$.

The next stage is to nail down the earlier assertion that the numbers with finite birthdays are precisely the dyadic rationals. Since the numbers are a field, there certainly are numbers of the form $m2^{-r}$ for integers $m$ and $r \ge 0$, so we just need to find them. To put it another way, we need to show that the provisional names we gave to numbers with finite birthdays are the correct ones.

**The Dyadic Rational Theorem.** *The Conway-Gonshor numbers with finite birthdays are precisely the dyadic rationals.*

*Proof.* We have the positive integers as the finite ordinals, and the negative integers as their negatives. It is clear that every other Conway-Gonshor number with a finite birthday lies between two earlier Conway-Gonshor numbers (one of which will have been born the day before). We now prove by induction that given integers $m$ and $r \geq -1$,

$$\{m2^{-r} \mid (m+1)2^{-r}\} = (2m+1)2^{-r-1}$$

If $2m+1$ is positive, $2m+1 = \{2m \mid \}$, whilst if it is negative, $2m+1 = \{ \mid 2(m+1)\}$. Either way $2m+1 = \{2m \mid 2(m+1)\}$. This is the $r = -1$ case of the theorem. We proceed by finite induction. For $r \geq 0$ let $x = \{m2^{-r} \mid (m+1)2^{-r}\}$, then

$$2x = x + x = \{x + m2^{-r} \mid x + (m+1)2^{-r}\}$$

However,

$$2m2^{-r} < x + m2^{-r} < (2m+1)2^{-r}, \text{ and } (2m+1)2^{-r} < x + (m+1)2^{-r} < (2m+2)2^{-r}$$

By induction $\{2m2^{-r} \mid (2m+2)2^{-r}\} = (2m+1)2^{-r}$, and therefore, since $2x$ lies in a narrower interval that includes $(2m+1)2^{-r}$, $2x = (2m+1)2^{-r}$, so $x = (2m+1)2^{-r-1}$. Now, since integers are of the form $m2^{-r}$ with $r = 0$, and every other number with a finite birthday is born between two earlier numbers, all numbers with finite birthdays are of the form $m2^{-r}$ with $r \geq 0$. Also any number of that form (with $m$ odd) is born exactly $r$ days after the second of the integers either side of it. ☑

Now let $\mathbb{R}$ be the standard real numbers. We shall assume all their usual properties. Let $\mathbb{D}$ denote the dyadic rationals, and we can identify the $\mathbb{D}$ inside $\mathbb{R}$ with the one inside **No**. Define a map $\psi : \mathbb{R} \to \mathbf{No}$, by

$$\psi(r) = \{d \in \mathbb{D} : d < r \mid d \in \mathbb{D} : d > r\}$$

Now if $r$ is not a dyadic rational, $\psi(r)$ is a number with birthday $\omega$ and its options are all the dyadic rationals, whilst if $r$ is a dyadic rational, $\psi(r)$ still has birthday $\omega$ and its options are all dyadic rationals except itself, and it has a finite conception day. Given the identification of dyadic rationals in $\mathbb{R}$ and **No**, $d < r$ iff $d < \psi(r)$ etc.

Now that we are dealing with numbers with infinitely many options, we need an alternative to the extra option theorem:

**The Cofinal Theorem.** *If $x$ and $y$ are numbers such that*

$$\forall x^L, \exists y^L > x^L, \text{ and } \forall y^L, \exists x^L > y^L$$
$$\forall x^R, \exists y^R < x^R, \text{ and } \forall y^R, \exists x^R < y^R$$

*Then $x = y$.*

The proof is immediate. Interlinked sets of options like this are called cofinal sets of options.

**The Real Number Theorem.** *$\psi$ as above is an order-preserving field monomorphism*

Ie it preserves order, addition and multiplication and is one-to-one.

*Proof.* $r < s \implies \exists d \in \mathbb{D}$ such that $r < d < s$. Then $d$ is a right option of $\psi(r)$ and a left option of $\psi(s)$, so $\psi(r) < \psi(s)$. This shows that $\psi$ is one-to-one and order preserving. Now,

$$(\psi(r) + d)^L = \psi(r)^L + d \vee \psi(r) + d^L$$

$\psi(r)^L + d$ takes on the values of all dyadic rationals less $r + d$, whilst $\psi(r) + d^L$ is cofinal with those dyadic rationals, and similarly for right options. Therefore $\psi(r) + d \in \psi(\mathbb{R})$, and $\psi(r) + d = \psi(r + d)$. $\psi(r) + \psi(s)$ has options of the form $\psi(r) + d$ and $d + \psi(s)$, and these are clearly cofinal with the options of $\psi(r) + \psi(s)$, ie $\psi(r + s) = \psi(r) + \psi(s)$.

Similarly $\psi$ preserves multiplication by dyadic rationals, because the $O$ options of $d\psi(r)$ are just $d$ times the $O$ options of $\psi(r)$. $\psi(r)\psi(s)$ has options of the form $d_r\psi(s) + \psi(r)d_s - d_r d_s = \psi(d_r s + r s_s - d_r d_s)$. So $\psi(r)\psi(r)$ has cofinal sets of dyadic options, and the argument proceeds as above to show that $\psi(r)\psi(r) \in \psi(\mathbb{R})$ and that $\psi$ preserves multiplication. ☑

So we have found the real numbers and the ordinals inside **No**, and from now on $\mathbb{D}$ and $\mathbb{R}$ will refer to the surreal dyadic rationals and reals.

## 6.2 Finishing Off Day $\omega$

We are not quite finished with day $\omega$. The reals that are not dyadic rationals were all numbers for which $L$ had no largest element, and $R$ has no smallest. So what about numbers such as

$$x \equiv \{\text{dyadic rationals} \leq 0 \mid \text{dyadic rationals} > 0\}$$

This number is bigger than 0, but smaller than any positive dyadic rational, and therefore smaller than any positive real number, and of course it is not in $\mathbb{R}$. It is our first infinitesimal. It turns out to be $1/\omega$, which we can prove by multiplying it by $\omega$. Choosing suitable forms, we have

$$\begin{aligned} \omega &= \{2^n \mid \}, \text{ for } n \in \mathbb{Z} \text{ with } n \geq 0 \\ x &= \{0 \mid 2^{-m}\}, \text{ for } m \in \mathbb{Z} \text{ with } m \geq 0 \\ \omega x &= \{2^n x + \omega 0 - 2^n 0 \mid 2^n x + 2^{-m}\omega - 2^{n-m}\} \end{aligned}$$

Now, $x$ is less than any positive real number, so $2^n x$ is still less that any positive real number (for $2^n x \geq r \implies x \geq 2^{-n} r$), so the left options of $\omega x$ are positive infinitesimals. In the same way the right options are all infinite. There is a huge interval between the left and right options, but the simplest number in that interval is 1, so $\omega x = 1$, and $x = 1/\omega$, as claimed. In fact on day $\omega$ we construct $d \pm 1/\omega, \forall d \in \mathbb{D}$.

Incidentally, the example of $1/3$ that we looked at earlier might suggest that options of $1/x$ constructed from longer sequences of options of $x$ are closer to $1/x$, but that is not necessarily true. Consider $\omega = \{\mathbb{N} \mid \}$. The sequence of options of length 0 gives us 0 as a left option, and the sequences of length 1 give us $1/n$ for $n \in \mathbb{N}$ as right options. That is enough to define $1/\omega$, and the later options are useless. The sequences of length 2 give left options of $1/n + 1/m - \omega/(nm)$ which is negative and infinite! In fact all the later options are infinite, negative for the left options and positive for the right ones.

There is in fact a cute way to translate between the binary expansion of a real number $r$ and its sign expansion, due to Knuth. The next theorem covers a little more than that.

**The Sign Expansion Theorem for $M_\omega$.**

*(a) The sign expansion of 0 is empty.*

*(b) If $r$ is a positive real number with $n \le r < n+1$ for some integer $n$, the sign expansion of $r$ starts with $n$ +s, (that's the plural of +), and stops there if $r = n$. If $r > n$, the next two signs are $+-$. The remaining signs follow the binary expansion of $r - n$, with 0 being translated as $-$ and 1 as $+$, except that if $r$ is a dyadic rational, the final 1 in the binary expansion is not translated.*

*(c) If $d$ is a positive dyadic rational, the sign expansion for $d + 1/\omega$ is that of $d$ followed by a $+$ and $\omega$ $-$s, whilst the the sign expansion for $d - 1/\omega$ is that of $d$ followed by a $-$ and $\omega$ $+$s.*

*(d) The sign expansion of omega is $\omega$ +s.*

*(e) The sign expansion of $-x$ is the sign expansion of $x$ with every $+$ changed into a $-$ and vice versa.*

*Proof.*

(a) 0 has no proper ancestors.

(b) The first $n$ ancestors are $0, 1, 2, 3 \cdots, n-1$, and these are all left ancestors, so the sign expansion starts with $n$ +s. If $r = n$ that is the end of the matter. Otherwise, $n < r < n+1$, and so $n$ is another left ancestor and $n+1$ a right ancestor, so the next two signs are $+-$. If $r = n + 1/2$, we add 1 to the binary expansion but the sign expansion is already finished, and we stop. If $n < r < n + 1/2$ we add 0 to the binary expansion and $-$ to the sign expansion whereas if $n + 1/2 < r < n + 1$ we add 1 to the binary expansion and $+$ to the sign expansion and carry on doing a binary search, which never finishes if $r$ is not a dyadic rational. For example

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 11/16 in binary | = | 0 | . | | 1 | | 0 | | 1 | | 1 |
| sign expansion | | | $+-$ | | $+$ | | $-$ | | $+$ | | |
| ancestry | | 0 | 1 | $\frac{1}{2}$ | | $\frac{3}{4}$ | | $\frac{5}{8}$ | | $\frac{11}{16}$ | |

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\pi$ in binary | = | 11 | . | | 0 | | 0 | | 1 | | 0 | | $\cdots$ | |
| sign expansion | | $+++$ | $+-$ | | $-$ | | $-$ | | $+$ | | $-$ | | $\cdots$ | |
| ancestry | | 0 1 2 3 | 4 | $3\frac{1}{2}$ | | $3\frac{1}{4}$ | | $3\frac{1}{8}$ | | $3\frac{3}{16}$ | | $3\frac{5}{32}$ | $\cdots$ | $\pi$ |

(c) If $r = d + 1/\omega$, we construct $d$ as above, but that is a left ancestor, so we add a $+$, and all the rest of the ancestors are right ancestors, so we have $\omega$ $-$s. Similarly when $r - d - 1/\omega$.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $1\frac{1}{2} + \frac{1}{\omega}$ in binary | = | 1 | . | | 1 | | | | | $+1/\omega$ | | | | |
| sign expansion | | $+$ | $+-$ | | $+$ | | $-$ | | $-$ | | $-$ | | $\cdots$ | |
| ancestry | | 0 1 | 2 | $1\frac{1}{2}$ | | $1\frac{3}{4}$ | | $1\frac{5}{8}$ | | $1\frac{9}{16}$ | | $1\frac{17}{32}$ | $\cdots$ | $1\frac{1}{2} + \frac{1}{\omega}$ |

(d) The ancestors are $\mathbb{N}$, and they are all left ancestors.

(e) $(-x)_\gamma \equiv -(x_\gamma)$, and $(-x)_\gamma < -x$ iff $(x_\gamma) > x$. ☑

Now that we have met our first infinite and infinitesimal numbers, it is time for a proper definition of these terms. Recall that $\mathbb{N}$ is the set of ordinals less that $\omega$.

**The Definition of Finite, Infinite and Infinitesimal.**

- *A number $x$ is finite iff $\exists n \in \mathbb{N}$ such that $-n < x < n$, and infinite otherwise.*
- *A non-zero number $x$ is infinitesimal iff $\forall d \in \mathbb{D}^+, -d < x < d$.*

Note that being finite is definitely not the same thing as being less that $\omega$, though of course that is true for ordinals. Not also that 0 does not count as an infinitesimal.

 

## 6.3 The Next Few Days

Since **No** is a field, it must contain such wonderful numbers as $\omega - 1$ as $\omega/2$, so it is time to start finding them. On day $\omega + 1$ we already know we get the ordinal $\omega + 1 \equiv \{\omega \mid \}$. However we also get the archetypal surreal number, the number that convinces you that you are in the presence of something truly original. Consider the number $x = \{\mathbb{N} \mid \omega\}$. To see what this is, add 1 to it

$$\begin{aligned} x &= \{\mathbb{N} \mid \omega\} \\ 1 &= \{0 \mid \} \\ x + 1 &= \{x, \mathbb{N} + 1 \mid \omega + 1\} \end{aligned}$$

The left options are less than $\omega$, but one of them is infinite, whilst the right option is greater than $\omega$, so by the simplicity theorem, $x + 1 = \omega$, which is the simplest infinite number of all, and therefore $x = \omega - 1$. In fact the positive infinite numbers produced on the first few infinite days are:

$$\begin{array}{lrl} \text{day } \omega & \omega & = \{\mathbb{N} \mid \} \\ \text{day } \omega+1 & \omega - 1 & = \{\mathbb{N} \mid \omega\} \\ & \omega + 1 & = \{\omega \mid \} \\ \text{day } \omega+2 & \omega - 2 & = \{\mathbb{N} \mid \omega - 1\} \\ & \omega - 1/2 & = \{\omega - 1 \mid \omega\} \\ & \omega + 1/2 & = \{\omega \mid \omega + 1\} \\ & \omega + 2 & = \{\omega + 1 \mid \} \end{array}$$

So the pattern of the first few days repeats itself, centred on $\omega$. After all the days $\omega + \mathbb{N}$ we have all the numbers $\omega + \mathbb{D}$. The next day is $\{\omega, \omega + 1, \omega + 2, \ldots\}$ which feels as though it ought to be $2\omega$. (In ordinal arithmetic is it $\omega 2$ but not $2\omega$, but we are not using ordinal arithmetic, but surreal arithmetic which is commutative, so we can stick to the more natural $2\omega$). It is worth proving that:

$$\begin{aligned} \omega &= \{0, 1, 2, \ldots \mid \} \\ \omega + \omega &= \{\omega + 0, \omega + 1, \omega + 2, \ldots \mid \} \end{aligned}$$

On this day, $2\omega$, there is a subtle change to this pattern. At the right hand end we get $2\omega$ itself of course, and in the middle we get $\omega + \mathbb{R}$ and $\omega + \mathbb{D} \pm 1/\omega$, just as we would expect. However, the leftmost number here is $x = \{\mathbb{N} \mid \omega - \mathbb{N}\}$ can hardly be $\omega - \omega$ because that is equal to 0. So what it it?

$$\begin{aligned} x &= \{\mathbb{N} \mid \omega - \mathbb{N}\} \\ x + x &= \{x + \mathbb{N} \mid x + \omega - \mathbb{N}\} \end{aligned}$$

Now $x < \omega - n$ for any $n \in \mathbb{N}$, so $x + n < \omega$, ie so all the left option of $x + x$ are less than $\omega$, but are infinite. Similarly, $x > n$, so $x + \omega - n > \omega$, ie all the right options of $x + x$ are bigger than $\omega$. Therefore $x + x = \omega$, and $x = \omega/2$.

Now let's turn to the interval between 0 and $\mathbb{D}^+$, the positive dyadic rationals. So far we have found $1/\omega$, born on day $\omega$. The next day we have two new numbers here, $x = \{0 \mid 1/\omega\}$ and $y = \{1/\omega \mid \mathbb{D}^+\}$.

$$x = \{0 \mid 1/\omega\}$$
$$x + x = \{x \mid x + 1/\omega\}$$

So the left option of $x + x$ is positive but less than $1/\omega$, whist the right option is greater than $1/\omega$ but still infinitesimal. So by the simplicity theorem, $x+x = 1/\omega$, the simplest positive infinitesimal, and therefore $x = 1/2\omega$.

$$\begin{aligned} 1/\omega &= \{0 \mid \mathbb{D}^+\} \\ 1/\omega + 1/\omega &= \{1/\omega \mid 1/\omega + \mathbb{D}^+\} \\ &= y, \text{ by the cofinal theorem} \end{aligned}$$

So $y = 2/\omega$. In fact the positive infinitesimal numbers produced on the first few infinite days are:

$$\begin{array}{lrl} \text{day } \omega & 1/\omega & = \{0 \mid \mathbb{D}^+\} \\ \text{day } \omega+1 & 1/2\omega & = \{0 \mid 1/\omega\} \\ & 2/\omega & = \{1/\omega \mid \mathbb{D}^+\} \\ \text{day } \omega+2 & 1/4\omega & = \{0 \mid 1/2\omega\} \\ & 3/4\omega & = \{1/2\omega \mid 1/\omega\} \\ & 3/2\omega & = \{1/\omega \mid 2/\omega\} \\ & 3/\omega & = \{2/\omega \mid \mathbb{D}^+\} \end{array}$$

So the pattern of the first few days repeats itself in a rather different way. After all the days $\omega + \mathbb{N}$ we have all the numbers $\mathbb{D}^+/\omega$ in the interval between 0 and $\mathbb{D}^+$, and similarly of course their negatives just to the left of 0. Again the pattern begins to break down on day $2\omega$. We have $1/\omega^2$ at the left hand end, and in the middle we have $\mathbb{R}/\omega$ and numbers like $\mathbb{D}^+/\omega \pm 1/\omega^2$. At the right hand end we have $x = \{\mathbb{D}^+/\omega \mid \mathbb{D}^+\}$

$$\begin{aligned} x &= \{\mathbb{D}^+/\omega \mid \mathbb{D}^+\} \\ x^2 &= \{xd/\omega + xe/\omega - de/\omega^2, xd + xe - de \mid xd/\omega + xe - de/\omega\} \end{aligned}$$

where $d$ and $e$ are dyadic rationals. Now

$$xd/\omega + xe/\omega - de/\omega^2 < xd/\omega + xe/\omega = (d+e)x/\omega < 1/\omega$$

So this sort of left option is positive but smaller that $1/\omega$, whilst the other sort of left option is negative. The right option is bigger than $1/\omega$ but still infinitesimal, so $x^2 = 1/\omega$, the simplest infinitesimal. So $x = 1/\sqrt{\omega}$.

## 6.4 The Square Root

Now that we have found a rather exotic square root, it is perhaps a good time to show that any positive number has a square root. Not only that, but there is a construction for it rather like the construction of the reciprocal, originally due to Clive Bach. As with the reciprocal, there is an obvious first guess, $\{\sqrt{x^L} \mid \sqrt{x^R}\}$, and as with the reciprocal, that is not good enough.

**The Definition of the Square Root.** *For a positive number $x$, define $\sqrt{x}$ as follows.*

$$\left(\sqrt{x}\right)^{\sum O_i} \equiv \frac{\prod\left(\sqrt{x^{O_i}} + \sqrt{x}\right) + \prod\left(\sqrt{x^{O_i}} - \sqrt{x}\right)}{\left(\prod\left(\sqrt{x^{O_i}} + \sqrt{x}\right) - \prod\left(\sqrt{x^{O_i}} - \sqrt{x}\right)\right) \Big/ \sqrt{x}}$$

 

*as $\mathcal{O}$ ranges over all non-empty finite sequences of non-negative options of $x$, except for those that consist entirely of an even number of 0's.*

As before, this means that every allowable non-empty finite sequence of non-negative options of $x$ defines a option of $\sqrt{x}$ according to this rule, and all options of $\sqrt{x}$ arise in this way. There are several things to observe about this definition before we start proving anything.

- A sequence with an odd number of left options generates a left option, and one with an even number of left options generates a right option.
- If we expand out the products all the appearances of $\sqrt{x}$ cancel out or are raised to an even power, so the definition is not circular.
- The denominator of the big fraction would be zero iff $\mathcal{O}$ consists entirely of an even number of 0's. The numerator is zero iff $\mathcal{O}$ consists entirely of an odd number of 0's.
- A sequence consisting of just one option $x^O$ generates an option $(\sqrt{x})^O = \sqrt{x^O}$, so this builds on our first guess.
- If $x \equiv \{1 \mid \} = 2$, our sequences are all just sequences of 1's, generating alternately the left and right options 1, 3/2, 7/5, 17/12, etc, which is promising, as these are the continued fraction convergents of $\sqrt{2}$.

**The Square Root Theorem.** *For $x > 0$, $\sqrt{x}$ as defined above is a number, and $\sqrt{x}^2 = x$.*

*Proof.* Let $\mathcal{O}$ and $\mathcal{P}$ be non-empty sequences of positive options of $x$, and let $g(\mathcal{O})$ be option of $\sqrt{x}$ that $\mathcal{O}$ constructs. Also let

$$\mathcal{O}_+ = \prod \left(\sqrt{x^{O_i}} + \sqrt{x}\right), \text{ and } \mathcal{O}_- = \prod \left(\sqrt{x^{O_i}} - \sqrt{x}\right)$$

Now,

$$g(\mathcal{O})^2 = x\frac{(\mathcal{O}_+)^2 + (\mathcal{O}_-)^2 + 2\prod\left(x^{O_i} - x\right)}{(\mathcal{O}_+)^2 + (\mathcal{O}_-)^2 - 2\prod\left(x^{O_i} - x\right)}$$

so $\left(\sqrt{x}^L\right)^2 < x < \left(\sqrt{x}^R\right)^2$, which means that $\sqrt{x}^L < \sqrt{x}^R$, and so $\sqrt{x}$ is a number. Now we need a little lemma. So long as $g(\mathcal{O})$ and $g(\mathcal{P})$ are not both 0,

$$\begin{aligned}
\frac{x + g(\mathcal{O})g(\mathcal{P})}{g(\mathcal{O}) + g(\mathcal{P})} &= \frac{(O_+ - O_-)(P_+ - P_-) + (O_+ + O_-)(P_+ + P_-)}{(O_+ + O_-)(P_+ - P_-) + (O_+ - O_-)(P_+ + P_-)} \\
&= \frac{O_+P_+ + O_-P_-}{O_+P_+ - O_-P_-} \\
&= g(\mathcal{O}\|\mathcal{P}), \text{ where } \| \text{ denotes the concatenation of the two sequences}
\end{aligned}$$

Now, looking at a typical option of $\sqrt{x}^2$,

$$\left(\sqrt{x}^2\right)^{L+O+P} = \sqrt{x}\sqrt{x}^O + \sqrt{x}\sqrt{x}^P - \sqrt{x}^O\sqrt{x}^P$$

If both options of $\sqrt{x}$ are zero, so is the resulting option of $\sqrt{x}^2$, and otherwise

$$\left(\sqrt{x}^2\right)^{L+O+P} - x = \left(\sqrt{x}^O + \sqrt{x}^P\right)\left(\sqrt{x} - \frac{x + \sqrt{x}^O\sqrt{x}^P}{\sqrt{x}^O + \sqrt{x}^P}\right)$$
$$= \left(\sqrt{x}^O + \sqrt{x}^P\right)\left(\sqrt{x} - \sqrt{x}^{O+P}\right)$$

So $\left(\sqrt{x}^2\right)^L - x < 0 < \left(\sqrt{x}^2\right)^R - x$, and this is still true when $\left(\sqrt{x}^2\right)^L = 0$. Meanwhile $\sqrt{x^L} < \sqrt{x} < \sqrt{x^R}$, so $\sqrt{x}^2 - x^R < 0 < \sqrt{x}^2 - x^L$.

Now the options of $\sqrt{x}^2 - x$ are of the form

$$\left(\sqrt{x}^2 - x\right)^O = \left(\sqrt{x}^2\right)^O - x \vee \sqrt{x}^2 - x^{L+O}$$

We have shown that both sorts of left option of $\sqrt{x}^2 - x$ are negative, and both sorts of right option are positive, so, by the simplicity theorem, $\sqrt{x}^2 - x = 0$ ☑

### 6.5 Subfields

**No** is a huge field, and it is interesting to look at some of its subfields. (In fact all ordered fields are subfields of **No**.) Of course we have the usual suspects, $\mathbb{Q}$ and $\mathbb{R}$, and fields in between such as $\mathbb{Q}(\sqrt{2})$, the splitting fields of bigger polynomials, and transcendental extensions such as $\mathbb{Q}(\pi)$. Another interesting field is $\mathbb{Q}(\omega)$. Algebraically, this is the same $\mathbb{Q}(\pi)$, since $\omega$ and $\pi$ are both transcendental over $\mathbb{Q}$. It is the order that distinguishes them: $\mathbb{Q}(\pi)$ is archemedean whereas $\mathbb{Q}(\omega)$ is not. We can also think about $\mathbb{Q}(\mathbf{Ord})$, which is $\mathbb{Q}$ extended with a proper class of transcendentals, but still doen't even have $\sqrt{2}$.

A completely different way of looking at this is to ask which $M_\alpha$ or $O_\alpha$ are subfields. Actually it is clear that $M_\alpha$ is never a subfield, because it never contains $\alpha + 1$. On the other hand, it seems pretty likely that $O_\alpha$ is a subfield if $\alpha$ is, for example, the first uncountable ordinal. As an example of a result that lies a little deeper than anything we have so far seen, and which we shall not prove, there is a cute theorem in [7] that says that

- $O_\alpha$ is an additive subgroup iff $\alpha = \omega^\gamma$ for some ordinal $\gamma$.
- $O_\alpha$ is a subring iff $\alpha = \omega^{\omega^\gamma}$ for some ordinal $\gamma$.
- $O_\alpha$ is a subfield iff $\alpha = \omega^\alpha$.

## 7 INTEGERS

The are many way to go from here, having established the basic structure. Here we take just one route, towards the analogue of integers, and we only explore them as far as giving us a better visualisation of the number field as a whole. Again it is Simon Norton who came up with the definition of what are called Omnific integers, henceforth just integers.

### 7.1 Definition and Elementary Properties

**The Definition of Integers.** *A number $x$ is an integer iff $x = \{x - 1 \mid x + 1\}$.*

We shall use the idea of the distance between two numbers $x$ and $y$, but distance here does not mean a real-valued metric, but is simply the non-negative surreal number $|x - y|$. We could think of this as a surreal valued metric, but we shall not take time out to explore the idea of surreal metric spaces! So an integer is just a number that is the simplest within a distance of 1 of itself. To show that this is a sensible definition, we need the following:

**The Integer Theorem.**

*(a) A finite number $x$ is an integer iff $x \in \mathbb{N}$ or $-x \in \mathbb{N}$.*

*(b) If $m$ and $n$ are integers, so are $n + m$, $-n$ and $nm$.*

*(c) For any number $x$, $\{x - 1 \mid x + 1\}$ is an integer.*

*Proof.*

(a) Clearly 0 is an integer. Otherwise, suppose without loss of generality, that $x > 0$. If $x \in \mathbb{N}$, $x = \{x - 1 \mid \} = \{x - 1 \mid x + 1\}$. On the other hand, if $x \notin \mathbb{N}$ then $x - 1 < \lfloor x \rfloor < x + 1$, but $\lfloor x \rfloor$ is simpler than $x$.

(b)
$$\begin{aligned}
n &= \{n - 1 \mid n + 1\} \\
m &= \{m - 1 \mid m + 1\}, \text{ so} \\
n + m &= \{(n - 1) + m, n + (m - 1) \mid (n + 1) + m \mid n + (m + 1)\} \\
&= \{(n + m) - 1 \mid (n + m) + 1\} \\
-n &= \{-(n + 1) \mid -(n - 1)\} \\
&= \{(-n) - 1 \mid (-n) + 1\}, \text{ and} \\
nm &= \{(n - 1)m + n(m - 1) - (n - 1)(m - 1), (n + 1)m + n(m + 1) - (n + 1)(m + 1) \\
&\qquad \mid (n - 1)m + n(m + 1) - (n - 1)(m + 1), (n + 1)m + n(m - 1) - (n + 1)(m - 1)\} \\
&= \{nm - 1 \mid nm + 1\}
\end{aligned}$$

(c) We can assume $x$ is not itself an integer. Consider the numbers $\{x - k \mid x + k\}$ for $k \in \mathbb{N}$. As the intervals gets wider, the numbers they define get, if anything, simpler. This can only happen finitely often, so $\exists c \in \mathbb{N}$ such that for $k \geq c$, $\{x - k \mid x + k\}$ has the same conception day as $\{x - c \mid x + c\}$, and must therefore be equal to it, else there would be a simpler number in between. Calling this number $x^*$, we have $x^* = \{x - c - 1 \mid x + c + 1\}$, but $x - c - 1 < x^* - 1$ and $x + c + 1 > x^* + 1$, so $x^* = \{x^* - 1 \mid x^* + 1\}$, and so $x^*$ is an integer. Now assuming w.l.o.g. that $x > x^*$, we have that $x^* + c - 1 < x < x^* + c$. $x^* + c - 1$ and $x^* + c$ are integers, so $x^* + c - 1 = \{x^* + c - 2 \mid x^* + c\}$, and $x^* + c = \{x^* + c - 1 \mid x^* + c + 1\}$, so one of them must the the simplest number between $x - 1$ and $x + 1$, ie $\{x - 1 \mid x + 1\}$, which is therefore an integer. ☑

Now the significance of the birthday addition theorem for these notes is

**The Integer Birthday Theorem.** *For integer $n$,*

*(a) Either $\beta(n + 1) = \beta(n) + 1$ or $\beta(n) = \beta(n + 1) + 1$.*

*(b) Either $\kappa(n + 1) = \kappa(n) + 1$ or $\kappa(n) = \kappa(n + 1) + 1$.*

*(c) Using $x^*$ as in the proof of the Integer Theorem, $\kappa(x^*)$ is a limit ordinal or 0.*

*Proof.*

(a) $\beta(n+1) \le \beta(n)+\beta(1) = \beta(n)+1$ and $\beta(n) \le \beta(n+1)+\beta(-1) = \beta(n+1)+(-1)$. However we cannot have $\beta(n) = \beta(n+1)$, for then there would be a simpler number between $n$ and $n+1$ which contradicts their being integers.

(b) Replace $\beta$ by $\kappa$ in the argument above.

(c) Suppose otherwise that $\kappa(x^*) = \gamma+1$ for some ordinal $\gamma$. Since $x^* = \{x^* - c \mid x^* + c\} \forall c \in \mathbb{N}$, $\widehat{x^*}$ is infinitely far from all its ancestors. Let $y$ be the $\gamma$-ancestor of $\widehat{x^*}$. Then by the Ancestor Theorem we can assume, without loss of generality, that $y$ is the greatest left ancestor of $\widehat{x^*}$. Now consider $y+1$, which is less than $x^*$. $\kappa(y+1) \le \gamma+1$. If $\kappa(y+1) \le \gamma$, then $y+1$ is equal to a greater left option of $\widehat{x^*}$ than $y$, whilst if If $\kappa(y+1) = \gamma+1$, there is a greater left option lying between $y+1$ and $\widehat{x^*}$. ☑

So now we know that integers come in "wedges", with some $x^*$ at the vertex having a limit (or 0) conception day $\gamma$, and then $x^* \pm k$ for $k \in \mathbb{N}$ with conception day $\gamma + k$. The numbers $x^*$ have several names in the literature because of their various interesting properties. For the moment, we shall call them stars. Every number lies between two integers, and so can be put into one if these wedges. The wedge whose vertex is $s$ then consists of all the numbers within a finite distance of $s$, or equivalently, all the numbers $x$ for which $x^* = s$. The wedges are open intervals, and equivalence classes of the relation $|x-y|$ is finite.

There is a cute characterisation of stars:

**The $\pm 2$ Theorem.** *$x$ is a star if and only iff $x = \{x-2 \mid x+2\}$.*

*Proof.* If $x$ is a star, $\beta(x+1) = \beta(x-1) = \beta(x)+1$, so $x$ is the simplest number between $x-2$ and $x-2$. If $x$ is an integer but not a star, one of $x-1$ and $x+1$ is simpler than $x$. If $x$ is not an integer, it is not even equal to $\{x-1 \mid x+1\}$. ☑

**The Star Theorem.** *If $x$ is a star, and $n \in \mathbb{N}$,*

*(a) $nx$ is a star;*

*(b) $x/n$ is a star;*

*(c) If $y/n \in \mathbb{N}$ for all $n \in \mathbb{N}$, then $y$ is a star.*

*Proof.*

(a)
$$\begin{aligned} x &= \{x-2 \mid x+2\} \\ n &= \{n-1 \mid \} \\ nx &= \{(n-1)x + n(x-2) - (n-1)(x-2) \mid (n-1)x + n(x+2) - (n-1)(x+2)\} \\ &= \{nx-2 \mid nx+2\} \end{aligned}$$

(b) Let $y = (x/n)^*$. $y$ is within a finite distance of $x/n$ and is a star. Therefore $ny$ is within a finite distance of $x$ and is a star, and therefore is $x$.

(c) $y$ is an integer, and so $|y-y^*|$ is an integer, say $n$. If $n > 0$, $y/2n$ and $y^*/2n$ are both integers, but $|y/2n - y^*/2n| = 1/2$ ☑

Conway calls stars divisible integers, because they are divisible by any finite integer. He shows, as we have now done, that any integer is uniquely the sum of a divisible integer and a finite integer.

Gonshor calls stars purely infinite numbers (even 0), and shows that any number is uniquely the sum of a purely infinite number, a real number and an infinitesimal number. That also follows from what we have shown.

## 7.2 The Sign Expansion of Integers

There is a nice characterisation of integers and stars in term of their sign expansions

**The Integer Sign Expansion Theorem.** *For any number $x$, $x$ is an integer iff $\sigma_\gamma(x) = \sigma_{\gamma+1}(x), \forall \gamma < \kappa(x)$. $x$ is a star iff in addition, $\kappa(x)$ is a limit ordinal.*

*Proof.* First we show the lemma that for any number $x$ and any $\gamma < \kappa(x)$, $x_{\gamma+1} \le x_\gamma + 1$. This is clearly true if $x < x_\gamma$, for then $x_{\gamma+1} < x_\gamma$. Suppose then that $x_\gamma < x$, so that $x_\gamma < x_{\gamma+1}$,

$$\begin{array}{ccc} x_{\gamma+1} & \le & x_\gamma + 1 \\ \{x_\gamma \mid ...\} & & \{x_\gamma, x_\gamma^L + 1 \mid (x_\gamma)^R + 1\} \end{array} \quad ?$$

We obviously do not have $x_\gamma + 1 \le x_\gamma$, and we cannot have $(x_\gamma)^R + 1 \le x_{\gamma+1}$, since any right option of $x_\gamma$ is also a right option of $x_{\gamma+1}$. That proves the lemma, and similarly, $x_{\gamma+1} \ge x_\gamma - 1$.

Now suppose that $\sigma_\gamma(x) = +$ and $\sigma_{\gamma+1}(x) = -$. Then $x_\gamma < x_{\gamma+2} < x_{\gamma+1}$, but $|x_{\gamma+1} - x_\gamma| \le 1$, so neither $x_{\gamma+2}$ nor any of its descendants can be integers.

Now suppose that $\sigma_\gamma(x) = \sigma_{\gamma+1}(x), \forall \gamma < \kappa(x)$. Then by induction $x_\gamma$ is an integer, $\forall \gamma < \kappa(x)$. We have two cases depending upon whether $\kappa(x)$ is a limit ordinal or not. If not, let $\alpha = \kappa(x) - 1$. Then $x_\alpha$ is an integer on some wedge. If it is the star of the wedge, both its successors are integers, one on each arm of the wedge, and $x$ must be one of them. If $x_\alpha$ is not a star, its successor moving away from the star is an integer, and this is $x$.

Finally we have the case where $\kappa(x)$ is a limit ordinal. Every ancestor of $x$ is on some wedge, but $x$ is not on that wedge. So $x$ is infinitely far from all its ancestors, so $\hat{x}$ is infinitely far from all its options, and is therefore an integer, and in fact a star. ☑

## 7.3 Back to the Future

We now know that the sign expansion of a star consists of runs of $\omega$ similar signs, and this provides the simplest way to see that the class of all stars is ordermorphic with **No** itself: construct the map which takes the sign expansion of a number, and repeats each sign $\omega$ times. This is another fractal view of **No**, but this time we zoom *out* to see the same pattern. Having zoomed out, we can zoom in again to fund that the class of stars between 0 and $\omega$ is ordermorphic with **No**. That's just how wrong the notion that $\omega$ is just beyond the end of the finite integers is: there is a proper class of stars between them, any two of which are infinitely far apart. And they are all in a linear order! Figure 1 illustrates this way of visualising the numbers, in their wedges, for the first few days.

We end with a flight of fancy that gets close to justifying playing some of my favourite film music to introduce my talk. As we move along the integers, instead on stopping neatly at the end on $\omega$ , we speed up, and when we reach 88 mph we take off, and land inconceivably far away. Each landing point (a star, or divisible integer), is the centre of a little finite world infinitely far from all the others. Here are a few more examples of numbers and their sign expansions, where, for example $+^\omega$ means a runs of $\omega$ $+s$:

$$
\begin{aligned}
5 &= +^5 \\
6 + 1/\omega &= +^7 -^\omega \\
\omega - 4 + 1/\omega &= +^\omega -^4 + -^\omega \\
\omega/3 + 1 &= +^\omega -^\omega (-^\omega +^\omega)^\omega + \\
1 + 1/\omega + 1/\omega^2 &= +^2 -^\omega + -^{\omega^2} \\
\omega^{2/3} &= +^\omega (-^{\omega^2} +^{\omega^2})^\omega
\end{aligned}
$$

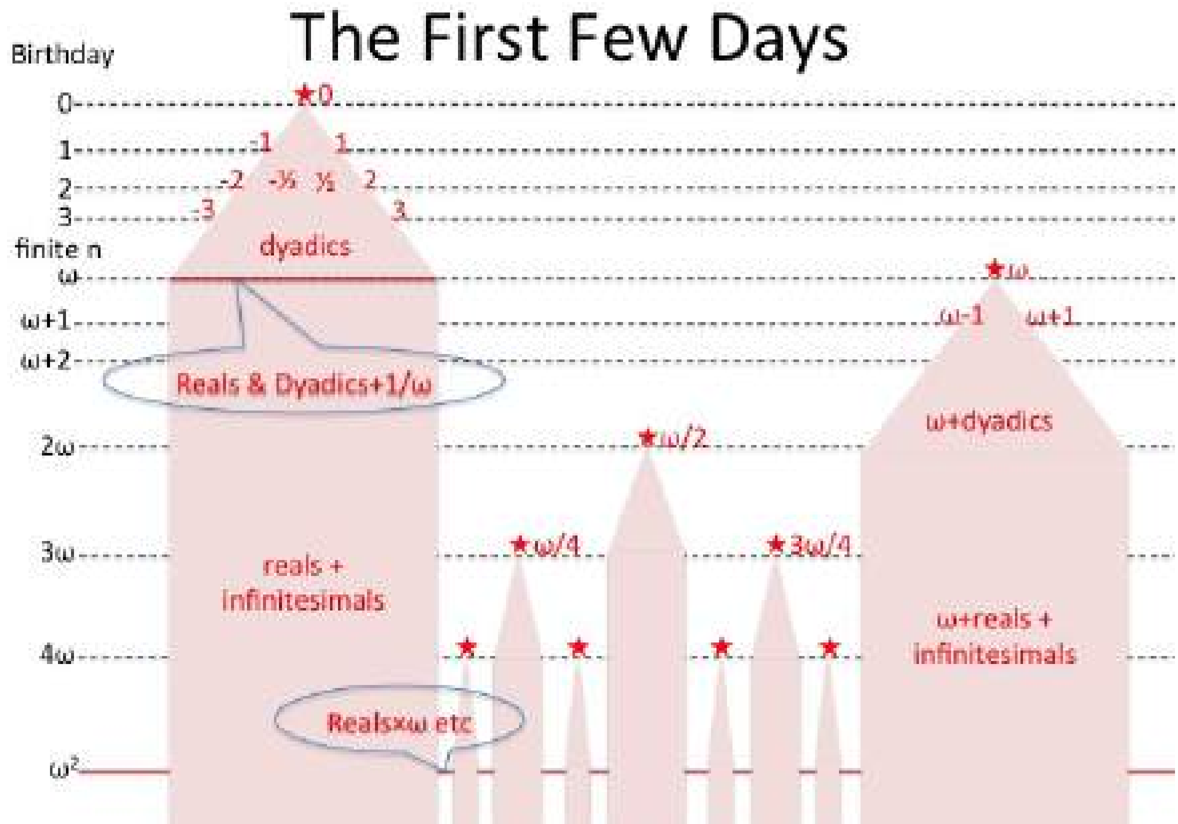


Figure 1: The First Few Days

## 8 FURTHER READING

To take this further, the best next stops are [4], [8], and [5]. They all take a different approach. As mentioned above, Gonshor takes the sign expansion as definitive, and this avoids the problem of having different numbers being equal. Alling, as in this paper, chooses a canonical form for each number, using an approach that he traces back to Cuesta Dutari in [6]. In our terms, his canonical form is closely related to $x_{\kappa(x)}$, the number that has all numbers older than

$x$ as options. It can be defined recursively as

$$\tilde{x} = \{\tilde{y}, y \in O_{\kappa x}, y < x \mid \tilde{y}, y \in O_{\kappa x}, y > x\}$$

These have a rather cumbersomely large set of options, eg his $3 \equiv \{-2, -1, -1/2, 0, 1/2, 1, 2 \mid \}$. $\{0, 1, 2 \mid \}$ seems neater. None of these authors use the conception day concept, which seems to me to simplify some of the arguments. (Conways talks about when a number is first born, which is the same idea.) None form $L$ and $R$ into a group, which dramatically shortens some of the proofs (Alling has essentially the same proof of the multiplication theorem as in this paper, but it takes 12 pages.)

The main piece of the basic theory we have not covered here is the normal form, or Conway name. Just as stars are the simplest members of the equivalence classes defined by the relation $|x-y|$ is finite, so Conway defines "leaders" to be the simplest members of the equivalence classes of positive numbers, defined by the relation $x/y$ is finite. These are also ordermorphic to the whole of **No**, and the ordermorphism is written $x \to \omega^x$ (this generalises ordinal exponentiation, in that $\omega^\alpha$, where $\alpha$ is an ordinal, means the same here as in standard ordinal arithmetic). This allows the construction of the normal form of a number. Any number $x$ can be expressed as

$$x = \sum_{\gamma<\alpha} \omega^{y_\gamma}.r_\gamma$$

where $\alpha$ is some ordinal, and $y_\gamma$ is a descending $\alpha$-length sequence of numbers, and the $r_\gamma$ are non-zero real numbers.

From there one can start doing something like analysis, despite there being no sensible topology on the surreal numbers. It can be shown that every positive number has an $n^{\text{th}}$ root for every $n \in \mathbb{N}$, and that every odd degree polynomial has a root (ie **No** is real closed). One can adjoin a square root of $-1$ to obtain the surcomplex numbers, which are algebraically closed. There are exponential and logarithmic functions which have the properties one would expect (except that surreal exponentiation does not agree with the $\omega^x$ function above). Alternatively one can go back to the integers and do some number theory. It is, for example, a theorem that every surreal number is "surrational" in the sense of being the ratio of two integers (for example $\pi = (\pi\omega)/\omega$.

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